A density result in Sobolev spaces

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Abstract We prove, for $1 \leq p < \infty$ and Ω a polygonal or regular open subset of $\mathbb{R}^N$, the density in $W^{1,p}(\Omega)$ of a set of regular functions satisfying a homogeneous Neumann condition on the boundary of Ω. We also give applications of this result and a generalization to mixed Dirichlet-Neumann boundary conditions.

1 Main results

1.1 Introduction

We consider, in the following, a bounded open subset Ω of $\mathbb{R}^N$ with a Lipschitz-continuous boundary (see [10]); we denote by σ the $(N-1)$-dimensional measure on $\partial\Omega$ and by $\mathbf{n}$ the unit normal to $\partial\Omega$ outward to Ω. For $p \in [1, \infty[$, $W^{1,p}(\Omega)$ is the usual Sobolev space.
Under such hypotheses, it is well-known (see e.g. [1]) that the space of restrictions to Ω of regular functions on $\mathbb{R}^N$ is dense in $W^{1,p}(\Omega)$. In this article, we intend to prove that, under additional hypotheses on Ω, the space of regular functions which satisfy a homogeneous Neumann condition on $\partial\Omega$, that is to say functions φ such that $\nabla\varphi \cdot \mathbf{n} = 0$ σ-a.e. on $\partial\Omega$, is also dense in $W^{1,p}(\Omega)$. The results we obtain are precisely stated in the two following subsections.
In Section 2, we consider the case when Ω is a polygonal open subset of $\mathbb{R}^N$; the proof strongly relies on the fact that $\partial\Omega$ is then made of pieces of hyperplanes. In Section 3, we prove the density result when Ω has a "regular" boundary; the proof, in this case, is based on a lemma which makes sure that, given a function g on $\partial\Omega$, we can find a function φ on $\mathbb{R}^N$ that is small in $W^{1,p}(\Omega)$ and such that $\nabla\varphi \cdot \mathbf{n} = g$ on $\partial\Omega$. Section 4 gives two applications of our main theorems, one to the weak formulation of the Neumann problem, the other to the convergence of a finite volume scheme; we shall also prove a generalization of the density result to mixed Dirichlet-Neumann boundary conditions.

1.2 The polygonal case

We assume first that Ω is a polygonal open subset of $\mathbb{R}^N$ (that is to say, Ω has a Lipschitz-continuous boundary and $\partial\Omega$ is contained in a finite union of hyperplanes); such open sets are of major interest when studying numerical schemes for partial differential equations (see [5] and Section 4.2). Our main result in this case is the following.

Theorem 1.1 *If Ω is a polygonal open bounded subset of $\mathbb{R}^N$ and $1 \leq p < \infty$, then*

$$\{\varphi_{|\Omega} \ : \ \varphi \in C_c^\infty(\mathbb{R}^N)\,,\ \nabla\varphi \cdot \mathbf{n} = 0 \ \sigma\text{-a.e. on } \partial\Omega\}$$

is dense in $W^{1,p}(\Omega)$.

Remark 1.1 *The Lipschitz-continuity hypothesis on $\partial\Omega$ is only used, in the proof of this theorem, to make sure of the density of regular functions in $W^{1,p}(\Omega)$. Thus, this theorem is also valid for open sets which are not Lipschitz-continuous in the sense of [10], but such that we can define a unit normal to $\partial\Omega$, such that the regular functions are dense in $W^{1,p}(\Omega)$ and which have a boundary contained in a finite union of hyperplanes (see [3] for an example of such a set).*

Notice that, though polygonal open sets are not regular, the singularities of their boundaries are very specific; this is what allows us to prove this theorem. Indeed, we cannot hope to obtain such a result for general open sets with Lipschitz-continuous boundary; the reason is the following: in dimension $N = 2$,

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we can construct an open set Ω such that the outer normal oscillates everywhere between two independent directions; thus, if φ is a regular function which satisfies $\nabla\varphi\cdot\mathbf{n}=0$ on $\partial\Omega$, the regular function $\nabla\varphi$, being orthogonal to $\mathbf{n}$ (which oscillates everywhere between two independent directions), must vanish on $\partial\Omega$; this means that φ must be constant on $\partial\Omega$ and that any limit of such regular functions is also constant on $\partial\Omega$.

Let us construct more precisely such a counter-example. We denote by $(s_n)_{n\geq 1}$ an enumeration of the rational numbers in $]-1,1[$ and take $\eta(s)=\sum_{n\geq 1}2^{-n-1}\sup(0,s-s_n)$; η is Lipschitz-continuous on $]-1,1[$ and its derivative is $\eta'(s)=\sum_{n\,|\,s>s_n}2^{-n-1}$. Let Ω be an open set of $\mathbb{R}^2$ with a Lipschitz-continuous boundary and such that $\Omega\cap]-1,1[^2$ is the epigraph of η.
Let $\varphi\in C^1(\mathbb{R}^N)$ satisfy $\nabla\varphi\cdot\mathbf{n}=0$ σ-a.e. on $\partial\Omega$. We have then, for a.e. $s\in]-1,1[$,

$$\eta'(s)\frac{\partial\varphi}{\partial x_1}(s,\eta(s))-\frac{\partial\varphi}{\partial x_2}(s,\eta(s))=0. \tag{1.1}$$

For all $n\geq 1$, we can approximate s_n from above and from below by sequences $(t_k^+)_{k\geq 1}$ and $(t_k^-)_{k\geq 1}$ of real numbers satisfying (1.1). We notice then that

$$\lim_{k\to\infty}\eta'(t_k^+)-\lim_{k\to\infty}\eta'(t_k^-)=2^{-n-1}$$

is not null; thus, passing to the limit in (1.1) applied to $(t_k^+)_{k\geq 1}$ and to $(t_k^-)_{k\geq 1}$, we obtain $\frac{\partial\varphi}{\partial x_1}(s_n,\eta(s_n))=0$. $(s_n)_{n\geq 1}$ being dense in $]-1,1[$, we deduce that the continuous function $\frac{\partial\varphi}{\partial x_1}(\cdot,\eta(\cdot))$ vanishes on $]-1,1[$ and, thanks to (1.1), that $\frac{\partial\varphi}{\partial x_2}(\cdot,\eta(\cdot))$ also vanishes on $]-1,1[$. Thus, $\varphi(\cdot,\eta(\cdot))$ is constant on $]-1,1[$.
Any limit in $W^{1,p}(\Omega)$ of functions $\varphi\in C^1(\mathbb{R}^N)$ satisfying $\nabla\varphi\cdot\mathbf{n}=0$ on $\partial\Omega$ is thus constant σ-a.e. on $\partial\Omega\cap]-1,1[^2$, and $\{\varphi_{|\Omega}\,:\,\varphi\in C^1(\mathbb{R}^N)\,,\ \nabla\varphi\cdot\mathbf{n}=0\ \sigma\text{-a.e. on }\partial\Omega\}$ cannot be dense in $W^{1,p}(\Omega)$.

1.3 The regular case

The preceding counter-example is based on the irregularity of the domain. In fact, assuming that the boundary of Ω is a bit more regular than just Lipschitz-continuous, we can prove a result similar to Theorem 1.1.

For $k\in\mathbb{N}$, we recall the usual defintion: a function is $C^{k,1}$-continuous if it is k-times continuously derivable and if its k-th derivative is Lipschitz-continuous. We say that a function is $C^{\infty,1}$-continuous if it is $C^{k,1}$-continuous for all $k\in\mathbb{N}$ ([2]).

Definition 1.1 *Let $k\in\mathbb{N}\cup\{\infty\}$. A bounded open subset Ω of $\mathbb{R}^N$ has a $C^{k,1}$-continuous boundary if, for all $a\in\partial\Omega$, there exists an orthonormal coordinate system $\mathcal{R}$ centered at a, an open set V of $\mathbb{R}^N$ containing a, such that $V=V'\times]-\alpha,\alpha[$ in $\mathcal{R}$, and a $C^{k,1}$-continuous function $\eta:V'\to]-\alpha,\alpha[$ such that, in $\mathcal{R}$, $\partial\Omega\cap V=\{(y',\eta(y'))\,,\ y'\in V'\}$ and $\Omega\cap V=\{(y',y_N)\in V\mid y_N>\eta(y')\}$.*

Notice that, if $k=0$, this definition corresponds to an open set with a Lipschitz-continuous boundary as in [10].
For $\varphi:\mathbb{R}^N\to\mathbb{R}$ and $a\in\partial\Omega$, we denote, if such a limit exists,

$$\frac{\partial\varphi}{\partial\mathbf{n}}(a)=\lim_{t\to 0}\frac{\varphi(a+t\mathbf{n}(a))-\varphi(a)}{t}.$$

Our second main result is the following.

[2] That is, the function is indefinitely derivable and all its derivatives are bounded.

Theorem 1.2 *Let $k \in \mathbb{N} \cup \{\infty\}$ and $1 \leq p < \infty$. If Ω is an open subset of $\mathbb{R}^N$ with a $C^{k+1,1}$-continuous boundary, then*

$$\left\{\varphi_{|\Omega} \;:\; \varphi \in C^{k,1}(\mathbb{R}^N)\,,\; \frac{\partial\varphi}{\partial\mathbf{n}} = 0 \text{ on } \partial\Omega\right\}$$

is dense in $W^{1,p}(\Omega)$.

Remark 1.2 *i) An alternate proof of this result can be found in [9]. However, the technique used in this reference (which consists in transporting the problem with well-chosen diffeomorphisms) can only be applied to open sets with at least $C^{2,1}$-continuous boundaries.*

ii) Notice the loss of a derivative: Ω has a $C^{k+1,1}$-continuous boundary, but we only obtain the density of $C^{k,1}$-continuous functions. This phenomenon can be correlated with the counter-example introduced in subsection 1.2.

iii) (Thierry Gallouët [6]) There is an alternate result to Theorem 1.2 which avoids this loss of a derivative: if Ω has a $C^{1,1}$-continuous boundary or is convex, then for all $u \in H^1(\Omega)$, there exists $(u_n)_{n\geq 1} \in H^2(\Omega)$ satisfying, for all $n \geq 1$, $\nabla u_n \cdot \mathbf{n} = 0$ σ-a.e. on $\partial\Omega$ and such that $u_n \to u$ in $H^1(\Omega)$.

The idea is to solve the following Neumann problem

$$\begin{cases} v_\varepsilon - \varepsilon\Delta v_\varepsilon = u & \text{in } \Omega, \\ \nabla v_\varepsilon \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Since Ω has a $C^{1,1}$-continuous boundary or is convex, the variational solution to this problem is in $H^2(\Omega)$ (see [7]); multiplying the equation by Δv_ε, we notice that $(v_\varepsilon)_{\varepsilon>0}$ is bounded in $H^1(\Omega)$ and that it weakly converges in this space to u; by Mazur's lemma, a convex combination of the $(v_\varepsilon)_{\varepsilon>0}$ strongly converges to u.

With this technique, we do not lose a derivative (we get the density of H^2 functions if Ω has a $C^{1,1}$-continuous boundary), in contrary to Theorem 1.2 (density of $C^{0,1}$-continuous functions under the same hypothesis); however, the derivatives are far less regular than in Theorem 1.2 (in L^2 instead of L^∞). Moreover, in the case of a polygonal open set, Theorem 1.1 gives a far better result than the method up above.

2 Proof of the density result in the polygonal case

Let Ω be a bounded polygonal open subset of $\mathbb{R}^N$; the boudary of Ω is made of vertices, edges, etc... up to pieces of hyperplanes; that is to say, there is a partition

$$\partial\Omega = \left(\bigsqcup_{i=1}^{l_0} F_i^0\right) \sqcup \cdots \sqcup \left(\bigsqcup_{i=1}^{l_{N-1}} F_i^{N-1}\right),$$

where each F_i^d is of dimension d ([3]).

The idea, to prove Theorem 1.1, is to approximate any regular function by functions that, on a neighborhood of each part F_i^k of $\partial\Omega$, only depend on the coordinates along this affine part (on a neighborhood of a vertex of $\partial\Omega$, the approximating functions will be constant; on a neighborhood of an edge of $\partial\Omega$, they

[3] The F_i^d can be formally defined the following way: we take $H_1, \ldots, H_{l_{N-1}}$ some pairwise disjoint affine hyperplanes, the union of which contains $\partial\Omega$ and such that, for all i, $H_i \cap \partial\Omega$ has dimension $N-1$; we define $(G_i^d)_{i\in[1,l_d]}$ as the non-empty intersections of $N-d$ distinct $(H_j \cap \partial\Omega)_{j\in[1,l_{N-1}]}$; F_i^d is then $G_i^d \backslash (\cup_{j=1}^{l_d-1} G_j^{d-1})$ if $d \geq 1$ and G_i^0 if $d = 0$. We leave the interested reader check on that this precise definition allows to justify the few intuitive facts we use about the F_i^k in the following proofs.

will only depend on the 1-dimensional coordinate along this edge; etc...). This approximation is done by induction: we first approximate a given regular function by functions that are constant on neighborhoods of the vertices of Ω; then we approximate a function which is constant on neighborhoods of the vertices by functions which, on neighborhoods of the edges of Ω, only depend on the coordinate along this edge; and so on...
This process is described by Lemma 2.1, which we state after having introduced a few notations.

If $d \in [0, N-1]$ and $i \in [1, l_d]$, P_i^d denotes the orthogonal projection on the affine space A_i^d generated by F_i^d.
For $d \in [0, N-1]$, we say that a function $u \in C_c^\infty(\mathbb{R}^N)$ satisfies the property $\mathcal{B}_d$ if, for all $i \in [1, l_d]$, $u = u \circ P_i^d$ on a neighborhood of F_i^d; this exactly means that, on a neighborhood of F_i^d, u only depends on the coordinates along F_i^d. To simplify the statement of the following lemma, we take as a convention that $\mathcal{B}_{-1}$ is the "empty" property, i.e. that any function in $C_c^\infty(\mathbb{R}^N)$ satisfies $\mathcal{B}_{-1}$.

Lemma 2.1 *Let $p \in [1, \infty[$ and $d \in [-1, N-2]$. If $u \in C_c^\infty(\mathbb{R}^N)$ satisfies $\mathcal{B}_d$, then there exists a sequence of functions $u_n \in C_c^\infty(\mathbb{R}^N)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$ and, for all $n \geq 1$, u_n satisfies $\mathcal{B}_{d+1}$. Moreover, if u has its support in the interior of some compact subset K of $\mathbb{R}^N$, then the functions $(u_n)_{n\geq 1}$ can be chosen with supports in the interior of K.*

Remark 2.1 *The additional conclusion concerning the supports of u and u_n will be useful in Section 4.3.*

Proof of Lemma 2.1
We denote by $B_N(\delta)$ the Euclidean open ball in $\mathbb{R}^N$ of center 0 and radius δ.
Before beginning the proof, we make two easy remarks (immediate consequences of the definition of an orthogonal projection):

$$\text{If } \overline{F_j^m} \subset \overline{F_i^k} \text{ and } x \in F_j^m + B_N(\delta), \text{ then } P_i^k(x) \in F_j^m + B_N(\delta). \tag{2.1}$$

$$\text{If } \overline{F_j^m} \subset \overline{F_i^k}, \text{ then } P_j^m \circ P_i^k = P_i^k \circ P_j^m = P_j^m. \tag{2.2}$$

Let us now prove the lemma. We take $u \in C^\infty(\mathbb{R}^N)$ with support in the interior of some compact subset K of $\mathbb{R}^N$ and satisfying $\mathcal{B}_d$; we will construct a sequence of functions $u_n \in C^\infty(\mathbb{R}^N)$ with supports in the interior of K and satisfying $\mathcal{B}_{d+1}$.

Step 1: we define a sequence of functions $(v_n)_{n\geq 1}$ which are, up to a localization in K, $(u - u_n)_{n\geq 1}$.
If $d \geq 0$, we take $\delta > 0$ such that, for all $l \in [1, l_d]$, $u = u \circ P_l^d$ on $F_l^d + B_N(\delta)$. If $d = -1$, the choice of δ does not matter (we take for example $\delta = 1$).
Let $i \in [1, l_{d+1}]$. F_i^{d+1} is linked to the others $(F_j^{d+1})_{j\neq i}$ via the $(F_l^d)_l$; that is, for all $j \neq i$, there exists $l \in [1, l_d]$ such that $\overline{F_i^{d+1}} \cap \overline{F_j^{d+1}} \subset \overline{F_l^d} \subset \overline{F_i^{d+1}}$ (with the convention $F_l^{-1} = \emptyset$). Thus, if we denote by J_i the set of l such that $\overline{F_l^d} \subset \overline{F_i^{d+1}}$ ([4]), the distance $\delta_{i,j}$ between $F_i^{d+1} \backslash (\cup_{l\in J_i} F_l^d + B_N(\delta/2))$ and F_j^{d+1} is positive (for $j \neq i$).
We define $\delta_0 = \inf(\delta, \inf_{i\neq j} \delta_{i,j})$ and we take, for all $i \in [1, l_{d+1}]$, $\theta_i \in C_c^\infty\left(F_i^{d+1} + B_N(\delta_0/2)\right)$ such that $\theta_i \equiv 1$ on $F_i^{d+1} + B_N(\delta_0/4)$.
Let $\gamma \in C_c^\infty(]-2,2[)$ such that $\gamma \equiv 1$ on $]-1,1[$; we denote $\gamma_n(t) = \gamma(nt)$ (notice that $\gamma_n(|\cdot|)$ is C^∞-continuous, since γ_n is constant on a neighborhood of 0); denoting by Id the identity mapping of $\mathbb{R}^N$, we let

$$v_n = \sum_{i=1}^{l_{d+1}} \theta_i \gamma_n\left(\left|Id - P_i^{d+1}\right|\right)\left(u - u \circ P_i^{d+1}\right) \in C_c^\infty(\mathbb{R}^N).$$

[4] J_i is also the set such that $\cup_{l\in J_i} \overline{F_l^d}$ is the boundary of F_i^{d+1} relatively to A_i^{d+1}

Step 2: we prove that $v_n \to 0$ in $W^{1,p}(\Omega)$ as $n \to \infty$.
We have, as $n \to \infty$ and for all $i \in [1, l_{d+1}]$, $\gamma_n(|x - P_i^{d+1}(x)|) \to 0$ if $x \neq P_i^{d+1}(x)$, that is to say if x does not belong to the space A_i^{d+1}; this space being of null measure (it is of dimension $d+1 \leq N-1$), we deduce that $v_n \to 0$ a.e. on Ω. v_n being bounded on Ω uniformly with respect to n, the dominated convergence theorem shows that $v_n \to 0$ in $L^p(\Omega)$ as $n \to \infty$.
The gradient of v_n is the sum of

$$\sum_{i=1}^{l_{d+1}} \gamma_n \left(\left| Id - P_i^{d+1} \right| \right) \nabla \left(\theta_i \left(u - u \circ P_i^{d+1} \right) \right) \tag{2.3}$$

and

$$\sum_{i=1}^{l_{d+1}} \theta_i \left(u - u \circ P_i^{d+1} \right) \gamma_n' \left(\left| Id - P_i^{d+1} \right| \right) \nabla (|Id - P_i^{d+1}|). \tag{2.4}$$

By the same argument as before, the term (2.3) tends to 0 in $L^p(\Omega)$ as $n \to \infty$.
The function $|Id - P_i^{d+1}|$ being Lipschitz-continuous on $\mathbb{R}^N$, its gradient is bounded on $\mathbb{R}^N$ (in fact, we can see that it is bounded by 1). The norm, in $L^p(\Omega)$, of (2.4) is thus bounded by

$$\sum_{i=1}^{l_{d+1}} ||\theta_i||_{L^\infty(\Omega)} \left| \left| (u - u \circ P_i^{d+1}) \gamma_n' \left(\left| Id - P_i^{d+1} \right| \right) \right| \right|_{L^p(\Omega)}.$$

But $\gamma_n'(|x - P_i^{d+1}(x)|) = 0$ if $|x - P_i^{d+1}(x)| \geq 2/n$, that is to say if x does not belong to $A_i^{d+1} + B_N(2/n)$ (recall that $|x - P_i^{d+1}(x)|$ is the distance between x and A_i^{d+1}); thus, using the Lipschitz-continuity property of u and the estimate $||\gamma_n'||_{L^\infty(\mathbb{R})} \leq n||\gamma'||_{L^\infty(\mathbb{R})}$, we bound the norm in $L^p(\Omega)$ of (2.4) by

$$2||\gamma'||_{L^\infty(\mathbb{R})} \mathrm{lip}(u) \sum_{i=1}^{l_{d+1}} ||\theta_i||_{L^\infty(\Omega)} \mathrm{meas} \left(\Omega \cap \left(A_i^{d+1} + B_N(2/n) \right) \right)^{1/p}. \tag{2.5}$$

Since the measure of Ω is finite and $\cap_{n \geq 1}(A_i^{d+1} + B_N(2/n)) = A_i^{d+1}$ is a non-increasing intersection of null measure, (2.5) tends to 0 as $n \to \infty$.
Both terms (2.3) and (2.4) going to 0 in $L^p(\Omega)$ as $n \to \infty$, we deduce the desired convergence of $(v_n)_{n \geq 1}$ to 0 in $W^{1,p}(\Omega)$.

Step 3: study of v_n on a neighborhood of F_i^{d+1}.
Let $i \in [1, l_{d+1}]$ and $U_{i,n}$ be the open set $F_i^{d+1} + B_N(\inf(\delta_0/4, 1/n))$. If $x \in U_{i,n}$, then $|x - P_i^{d+1}(x)| = \mathrm{dist}(x, A_i^{d+1}) \leq \mathrm{dist}(x, F_i^{d+1}) < 1/n$, so that $\gamma_n(|x - P_i^{d+1}(x)|) = 1$. Thus, on $U_{i,n}$,

$$v_n = u - u \circ P_i^{d+1} + \sum_{j \neq i} \theta_j \gamma_n (|Id - P_j^{d+1}|)(u - u \circ P_j^{d+1}).$$

Let $j \neq i$ and $x \in U_{i,n}$ be such that $\theta_j(x) \neq 0$. We have then $x \in (F_i^{d+1} + B_N(\delta_0/4)) \cap (F_j^{d+1} + B_N(\delta_0/2))$; we write $x = z + h$ with $z \in F_i^{d+1}$ and $|h| < \delta_0/4$, so that $z \in F_i^{d+1} \cap (F_j^{d+1} + B_N(3\delta_0/4))$; by definition of $\delta_0 \leq \delta_{j,i}$, this implies $z \in \cup_{l \in J_j}(F_l^d + B_N(\delta/2))$, and thus (since $\delta_0 \leq \delta$), $x \in F_l^d + B_N(\delta)$ for some $l \in J_j$. By (2.1), and since $\overline{F_l^d} \subset \overline{F_j^{d+1}}$, we get then $(x, P_j^{d+1}(x)) \in (F_l^d + B_N(\delta))^2$, which gives, by definition of δ and by (2.2), $u(x) = u(P_l^d(x))$ and $u(P_j^{d+1}(x)) = u(P_l^d(P_j^{d+1}(x))) = u(P_l^d(x))$. We deduce then $u(x) - u(P_j^{d+1}(x)) = 0$.
Thus, on $U_{i,n}$, we have $v_n = u - u \circ P_i^{d+1}$.

Step 4: conclusion.
Let $\Theta \in C_c^\infty(\mathrm{int}(K))$ and $\varepsilon > 0$ such that $\Theta \equiv 1$ on $\mathrm{supp}(u) + B_N(\varepsilon)$.

Define $u_n = u - \Theta v_n \in C_c^\infty(\mathbb{R}^N)$; $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$, the support of u_n is contained in the interior of K and, for all $i \in [1, l_{d+1}]$, we have, on $U_{i,n}$,

$$u_n = u - \Theta u + \Theta(u \circ P_i^{d+1}) = (1 - \Theta)u + \Theta(u \circ P_i^{d+1}) = \Theta(u \circ P_i^{d+1})$$

(because $1 - \Theta \equiv 0$ on $\mathrm{supp}(u)$).
Let $i \in [1, l_{d+1}]$ and $\mathcal{U}_{i,n} = F_i^{d+1} + B_N(\inf(\delta_0/4, 1/n, \varepsilon)) \subset U_{i,n}$.

- If $x \in \mathcal{U}_{i,n} \cap (\mathrm{supp}(u) + B_N(\varepsilon))$, then $u_n(x) = \Theta(x)u(P_i^{d+1}(x)) = u(P_i^{d+1}(x))$.
- If $x \in \mathcal{U}_{i,n} \backslash (\mathrm{supp}(u) + B_N(\varepsilon)) \subset F_i^{d+1} + B_N(\varepsilon)$, we have $|x - P_i^{d+1}(x)| < \varepsilon$ ($|x - P_i^{d+1}(x)|$ is the distance between x and $A_i^{d+1} \supset F_i^{d+1}$), and thus $P_i^{d+1}(x) \notin \mathrm{supp}(u)$, which gives $u_n(x) = \Theta(x)u(P_i^{d+1}(x)) = 0 = u(P_i^{d+1}(x))$.

Thus, in either case,

$$u_n = u \circ P_i^{d+1} \quad \text{on} \quad \mathcal{U}_{i,n} = F_i^{d+1} + B_N(\inf(\delta_0/4, 1/n, \varepsilon)). \tag{2.6}$$

If $x \in \mathcal{U}_{i,n}$ then, by (2.1), $P_i^{d+1}(x) \in \mathcal{U}_{i,n}$, so that, by (2.6) and (2.2),

$$u_n(P_i^{d+1}(x)) = u(P_i^{d+1}(P_i^{d+1}(x))) = u(P_i^{d+1}(x)) = u_n(x).$$

Thus, $u_n = u_n \circ P_i^{d+1}$ on a neighborhood of F_i^{d+1}, and the lemma is proved. ■

Proof of Theorem 1.1
Ω having a Lipschitz-continuous boundary, the space of the restrictions to Ω of functions in $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\Omega)$. An easy induction based on Lemma 2.1 allows to see then that the space

$$E = \{\varphi_{|\Omega} \ : \varphi \in C_c^\infty(\mathbb{R}^N)\,,\ \varphi = \varphi \circ P_i^{N-1} \text{ on a neighborhood of } F_i^{N-1}\ (\forall i \in [1, l_{N-1}])\}$$

is dense in $W^{1,p}(\Omega)$.
To prove Theorem 1.1, we just need to prove that the functions in E satisfy the homogeneous Neumann boundary condition on $\partial\Omega$.
But, if $\varphi \in E$, we have, for all $i \in [1, l_{N-1}]$, on a neighborhood of F_i^{N-1}, $\nabla\varphi = L_i^{N-1}\nabla\varphi \circ P_i^{N-1}$, where L_i^{N-1} is the transpose of the linear part of P_i^{N-1}, that is, L_i^{N-1} is the orthogonal projection on the vector space V_i^{N-1} parallel to A_i^{N-1} (an orthogonal projection is always self-adjoint). On a neighborhood of F_i^{N-1}, the gradient of φ is thus an element of V_i^{N-1}.
But it is quite easy to see that, on F_i^{N-1}, the unit normal to $\partial\Omega$ outward to Ω is defined and orthogonal to V_i^{N-1} (because F_i^{N-1} is a $(N-1)$-dimensional piece of $\partial\Omega$).
Thus, $\nabla\varphi \cdot \mathbf{n} = 0$ on F_i^{N-1} for all $i \in [1, l_{N-1}]$. Since $\cup_{i\in[1,l_{N-1}]} F_i^{N-1}$ covers $\partial\Omega$ up to a set of null σ-measure (the remaining set is of dimension $N-2$), this concludes the proof of the theorem. ■

Remark 2.2 *If Ω is a bounded open set with singularities of the same kind as the singularities of polygonal open sets, a result similar to Theorem 1.1 can be proved for Ω.*
For example, if there exists, locally, a $C^{r,1}$-diffeomorphism ($r \geq 1$) which preserves the outer normal ([5]) and tranforms the singularities of Ω into the singularities of a polygonal open set, we can prove the density in $W^{1,p}(\Omega)$ of

$$\{\varphi_{|\Omega} \ :\ \varphi \in C^{r,1}(\mathbb{R}^N)\,,\ \nabla\varphi \cdot \mathbf{n} = 0\ \sigma\text{-a.e. on } \Omega.\}.$$

This gives in fact another proof (the one in [9]) of Theorem 1.2, but only for $k \geq 1$.
A crucial example of this is $\Omega = O \times]0, T[$, where O is an open set of $\mathbb{R}^{N-1}$ with a $C^{r+1,1}$-continuous boundary. Though Ω has a boundary which is only Lipschitz-continuous, the singularities of this boundary are, up to a $C^{r,1}$-diffeomorphism, similar to the singularities of a polygonal open set.

[5] Such diffeomorphisms can be constructed, for example, thanks to the flow of a vector field which is, on $\partial\Omega$, equal to the unit normal.

3 Proof of the density result in the regular case

We assume here that Ω is a bounded open set with a $C^{k+1,1}$-continuous boundary (for a $k \in \mathbb{N} \cup \{\infty\}$). The regular functions being dense in $W^{1,p}(\Omega)$, Theorem 1.2 is an immediate consequence of the following proposition. We state this proposition to show that, as in the polygonal case, our technique of approximation is local (that is, the supports of the approximating functions are not far from the support of the function to approximate).

Proposition 3.1 *Let $p \in [1, \infty[$, $k \in \mathbb{N} \cup \{\infty\}$ and Ω be an open set of $\mathbb{R}^N$ with a $C^{k+1,1}$-continuous boundary. If $u \in C^\infty(\mathbb{R}^N)$, then there exists a sequence of functions $(u_n)_{n\geq 1} \in C^{k,1}(\mathbb{R}^N)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$ as $n \to \infty$ and, for all $n \geq 1$ and all $a \in \partial\Omega$, $\frac{\partial u_n}{\partial \mathbf{n}}(a) = 0$.*
Moreover, if u has its support in the interior of some compact subset K of $\mathbb{R}^N$, then the functions $(u_n)_{n\geq 1}$ can be chosen with supports in the interior of K.

Remark 3.1 *i) If $k = 0$, it is not true that, for a given $\varphi \in C^{k,1}(\mathbb{R}^N)$, $\frac{\partial \varphi}{\partial \mathbf{n}}(a)$ exists for some $a \in \partial\Omega$, let alone for all $a \in \partial\Omega$. We shall however see that the sequence $(u_n)_{n\geq 1}$ is really such that $\frac{\partial u_n}{\partial \mathbf{n}}(a)$ exists and is null* for all $a \in \partial\Omega$. *Notice that this is what we have announced in Theorem 1.2.*

ii) A close examination of the proofs which follow also shows that, if Ω has a Lipschitz-continuous boundary which is $C^{k+1,1}$-continuous "in an open set O"([6]), then we can approximate a given function in $W^{1,p}(\Omega)$ by $C^{k,1}$-continuous functions which satisfy a homogeneous Neumann boundary condition on $\partial\Omega \cap O$. Although the problem is not easily localizable (if φ satisfies a homogeneous Neumann boundary condition and γ is a regular function, $\gamma\varphi$ does not necessarily satisfies a homogeneous Neumann boundary condition), our techniques are however local ones.

The idea to prove Proposition 3.1 is the following: the function $g = \nabla u \cdot \mathbf{n}$ is $C^{k,1}$-continuous on $\partial\Omega$; we construct a sequence $(\gamma_n)_{n\geq 1} \in C^{k,1}(\mathbb{R}^N)$ which converges to 0 in $W^{1,p}(\Omega)$ and such that, for all $n \geq 1$, $\frac{\partial \gamma_n}{\partial \mathbf{n}} = g$ on $\partial\Omega$; the sequence $u_n = u - \gamma_n$ converges then to u in $W^{1,p}(\Omega)$ and satisfies the homogeneous Neumann boundary condition on $\partial\Omega$.
The only difficulty in this proof is the construction of the sequence $(\gamma_n)_{n\geq 1}$.

The first lemma, which states the existence of a local projection on $\partial\Omega$, is quite classical when $\partial\Omega$ is a regular submanifold of $\mathbb{R}^N$. We however prove it completely because, when $\partial\Omega$ is only $C^{1,1}$-continuous, the main tool of the proof is not so usual.

Lemma 3.1 *Let $k \in \mathbb{N} \cup \{\infty\}$ and Ω be an open set of $\mathbb{R}^N$ with a $C^{k+1,1}$-continuous boundary. There exists an open set U of $\mathbb{R}^N$ containing $\partial\Omega$ and a $C^{k,1}$-continuous application $P : U \to \partial\Omega$ such that*

i) for all $y \in U$, $P(y)$ is the unique $x \in \partial\Omega$ satisfying $\mathrm{dist}(y, \partial\Omega) = |y - x|$,

ii) for all $a \in \partial\Omega$, there exists $t_a > 0$ such that, for all $|t| < t_a$, $P(a + t\mathbf{n}(a)) = a$,

iii) for all $y \in U \backslash \partial\Omega$, $\mathbf{n}(P(y)) \cdot (y - P(y)) \neq 0$.

Proof of Lemma 3.1
Step 1: local construction.
We prove in this step that, for all $a \in \partial\Omega$, there exists an open set U_a of $\mathbb{R}^N$ containing a and a $C^{k,1}$-continuous application $P_a : U_a \to \partial\Omega$ such that, for all $y \in U_a$, $P_a(y)$ is the unique $x \in \partial\Omega$ satisfying $\mathrm{dist}(y, \partial\Omega) = |y - x|$.

[6] That is to say, Definition 1.1 holds for all $a \in \partial\Omega \cap O$.

Let $a \in \partial\Omega$ and take $\mathcal{R}$, $V = V' \times]-\alpha, \alpha[$ and $\eta : V' \to]-\alpha, \alpha[$ given for a by Definition 1.1. From now on in this step, we use $\mathcal{R}$ as a system for all our coordinates (notice that the norm and the distance are not modified by this change of coordinates).

Let us first study, for a given $y = (y', y_N)$, the solutions x' to $x' - y' + (\eta(x') - y_N)\nabla\eta(x') = 0$.
$F(x', y) = x' - y' + (\eta(x') - y_N)\nabla\eta(x')$ is $C^{k,1}$-continuous on $V' \times \mathbb{R}^N$ and is null at $(x', y) = (0, 0)$. Moreover, when it exists, $\frac{\partial F}{\partial x'}(x', y) = Id + \nabla\eta(x')\nabla\eta(x')^T + (\eta(x') - y_N)\eta''(x')$ (where $\eta''(x')$ is confused with the Hessian matrix of η).
If $k \geq 1$ then, F being C^k-continuous and $\frac{\partial F}{\partial x'}(0, 0) = Id + \nabla\eta(0)\nabla\eta(0)^T$ being definite positive, the classical implicit function theorem gives an open set $W \subset V'$ of $\mathbb{R}^{N-1}$ containing 0, an open set U of $\mathbb{R}^N$ containing 0 and a C^k-continuous application $f : U \to W$ such that, for all $(x', y) \in W \times U$, $F(x', y) = 0$ if and only if $x' = f(y)$. Moreover, since $f'(y) = -\left(\frac{\partial F}{\partial x'}(f(y), y)\right)^{-1} \circ \frac{\partial F}{\partial y}(f(y), y)$ and F is $C^{k,1}$-continuous, f is in fact $C^{k,1}$-continuous (we reduce U if necessary).
If $k = 0$, then $\nabla\eta$ is Lipschitz-continuous on V'; there exists thus $C > 0$ such that, for all $x' \in V'$, if $\eta''(x')$ exists, then $||\eta''(x')|| \leq C$ ($||\cdot||$ denotes the norm on the space of $(N-1) \times (N-1)$ matrices induced by the Euclidean norm in $\mathbb{R}^{N-1}$). Thus, for all $\xi \in \mathbb{R}^{N-1}$, if (x', y) is such that $\frac{\partial F}{\partial x'}(x', y)$ exists, we have

$$\frac{\partial F}{\partial x'}(x', y)\xi \cdot \xi \geq |\xi|^2 + (\nabla\eta(x')^T\xi)^2 - C|\eta(x') - y_N||\xi|^2.$$

Assuming that $(x', y) \to (0, 0)$ and that $\lim_{(x',y)\to(0,0)} \frac{\partial F}{\partial x'}(x', y)$ exists, this inequality lets us see that $\lim_{(x',y)\to(0,0)} \frac{\partial F}{\partial x'}(x', y)$ is a 1-coercive matrix (that is to say a $(N-1) \times (N-1)$ matrix A such that, for all $\xi \in \mathbb{R}^{N-1}$, $A\xi \cdot \xi \geq |\xi|^2$). Thus, any convex combination of such limits is also 1-coercive; denoting by S the set of $(x', y) \in V' \times \mathbb{R}^N$ such that F is differentiable with respect to x' at (x', y), we have thus proven that the set

$$\text{co}\left\{\lim_{(x',y)\to(0,0)} \frac{\partial F}{\partial x'}(x', y),\ (x', y) \in S\right\}$$

is made of invertible matrices. The Lipschitz implicit function theorem of [2] gives then an open set $W \subset V'$ of $\mathbb{R}^{N-1}$ containing 0, an open set U of $\mathbb{R}^N$ containing 0 and a Lipschitz-continuous application $f : U \to W$ such that, for all $(x', y) \in W \times U$, $F(x', y) = 0$ if and only if $x' = f(y)$.

Let $\beta > 0$ such that $B_N(\beta) \subset V$, $B_{N-1}(\beta) \subset W$ and $B_N(\beta/2) \subset U$; let $y \in B_N(\beta/2)$. By compacity of $\partial\Omega$, there exists some points in $\partial\Omega$ that are at distance $\text{dist}(y, \partial\Omega)$ of y. Moreover, since $0 \in \partial\Omega$, if x is such a point, we have $|x| \leq |y| + |x - y| \leq |y| + |y - 0| < \beta$, that is to say $x \in B_N(\beta) \subset V$.
x can thus be written as $(x', \eta(x'))$ for a $x' \in B_{N-1}(\beta) \subset W$; x' is then a minimum of the C^1-continuous function $|\cdot - y'|^2 + |\eta(\cdot) - y_N|^2$ on V' and we deduce that $x' - y' + (\eta(x') - y_N)\nabla\eta(x') = 0$.
Since $(x', y) \in W \times U$, x' is unique and $x' = f(y)$ (f has been constructed up above).
There can thus be only one projection of y on $\partial\Omega$; it is given by a function of y which is $C^{k,1}$-continuous on $B_N(\delta/2)$. This concludes this step (with $U_a = B_N(\delta/2)$ and $P_a(y) = (f(y), \eta(f(y)))$ in $\mathcal{R}$).

Step 2: we cover the compact set $\partial\Omega$ by a finite number of U_{a_i}, $i = 1, \ldots, l$, constructed in step 1.
$\cup_{i=1}^l U_{a_i}$ being an open set of $\mathbb{R}^N$ containing $\partial\Omega$, there exists an open set U of $\mathbb{R}^N$ containing $\partial\Omega$ and relatively compact in $\cup_{i=1}^l U_{a_i}$. Define $P : U \to \partial\Omega$ by: $\forall y \in U$, $P(y)$ is the unique point of $\partial\Omega$ at distance $\text{dist}(y, \partial\Omega)$ of y (since $y \in U_{a_i}$ for a certain $i \in [1, l]$, we know that such a point exists and is unique).
By construction of P and of the $(P_{a_i})_{i\in[1,l]}$, and by uniqueness of the point at distance $\text{dist}(y, \partial\Omega)$ of y when $y \in U$, we have $P = P_{a_i}$ on $U_{a_i} \cap U$. P is thus $C^{k,1}$-continuous on U.

Let us now check that, for all $a \in \partial\Omega$ and t small enough, we have $P(a + t\mathbf{n}(a)) = a$. Since the projection of a point of U on $\partial\Omega$ is unique, we have, on a neighborhood of a, $P = P_a$. By the study made in step 1, and using the notations introduced in this step (in which case the expression of $\mathbf{n}(a)$ is $(\sqrt{1 + |\nabla\eta(0)|^2})^{-1}(\nabla\eta(0), -1)^T)$, we see that, for t small enough, $P(a + t\mathbf{n}(a)) = (x', \eta(x'))$ where x' is the unique solution on a neighborhood of 0 to $x' - t(\sqrt{1 + |\nabla\eta(0)|^2})^{-1}\nabla\eta(0) + (\eta(x') +$

$t(\sqrt{1+|\nabla\eta(0)|^2})^{-1})\nabla\eta(x')=0$; but $x'=0$ is a solution to this equation. This means that $P(a+t\mathbf{n}(a)) = (0,\eta(0))=0$ in $\mathcal{R}$, that is to say $P(a+t\mathbf{n}(a))=a$.

To conclude this proof, it remains to see that the open set U given above satisfies item iii) of the lemma. Let $y\in U$; there exists $i\in[1,l]$ such that $y\in U_{a_i}$; by the study made in step 1, and with the same notations, we have $P(y)=(x',\eta(x'))$ where $x'\in V'$ satisfies $x'-y'+(\eta(x')-y_N)\nabla\eta(x')=0$. If $\mathbf{n}(P(y))\cdot(y-P(y))=0$, then $(\nabla\eta(x'),-1)^T\cdot(y'-x',y_N-\eta(x'))^T=0$ (because $\mathbf{n}(P(y))=(\sqrt{1+|\nabla\eta(x')|^2})^{-1}(\nabla\eta(x'),-1))$, so that $(y'-x')\cdot\nabla\eta(x')-(y_N-\eta(x'))=0$. By using the equation satisfied by x', we deduce that $(\eta(x')-y_N)(|\nabla\eta(x')|^2+1)=0$, that is to say $y_N=\eta(x')$ and, once again thanks to the equation satisfied by x', $x'=y'$. This gives $y=P(y)\in\partial\Omega$.
Thus, if $y\in U\backslash\partial\Omega$, we have $\mathbf{n}(P(y))\cdot(y-P(y))\neq 0$. ■

The following lemma gives the existence of the $(\gamma_n)_{n\geq 1}$ needed in the proof of Proposition 3.1.

Lemma 3.2 *Let $p\in[1,+\infty[$, $k\in\mathbb{N}\cup\{\infty\}$ and Ω be an open set of $\mathbb{R}^N$ with a $C^{k+1,1}$-continuous boundary. If $g\in C^{k,1}(\mathbb{R}^N)$ has its support in the interior of a compact subset K of $\mathbb{R}^N$, then for all $\varepsilon>0$ there exists $\gamma\in C^{k,1}(\mathbb{R}^N)$ with support in the interior of K such that $||\gamma||_{W^{1,p}(\Omega)}<\varepsilon$ and, for all $a\in\partial\Omega$, $\frac{\partial\gamma}{\partial\mathbf{n}}(a)=g(a)$.*

Proof of Lemma 3.2
Let U and P given by Lemma 3.1; we can suppose that U is bounded. Let $\theta\in C_c^\infty(\mathrm{int}(K)\cap U)$ such that $\theta\equiv 1$ on a neighborhood of $\mathrm{supp}(g)\cap\partial\Omega$.
Let $h\in C_c^\infty(]-1,1[)$ such that $h(0)=0$ and $h'(0)=1$; for $\delta>0$, we denote $h_\delta(x)=\delta h(x/\delta)$.
Define $\gamma_\delta(y)=\theta(y)g(P(y))h_\delta(\mathbf{n}(P(y))\cdot(y-P(y)))$; this function is well defined and $C^{k,1}$-continuous on U; since its support is a compact subset of $\mathrm{int}(K)\cap U$, its extension to $\mathbb{R}^N$ by 0 outside U is in $C^{k,1}(\mathbb{R}^N)$ and has a compact support in the interior of K.

Let us first check that, for all $a\in\partial\Omega$, $\frac{\partial\gamma_\delta}{\partial\mathbf{n}}(a)$ exists and is equal to $g(a)$. We study different cases, depending on the position of a on $\partial\Omega$.

- If $a\in\partial\Omega\backslash K$, the result is quite clear because, for t small enough, $(a,a+t\mathbf{n}(a))\notin\mathrm{supp}(\theta)$ so that $\gamma_\delta(a+t\mathbf{n}(a))=\gamma_\delta(a)=0=g(a)$.
- If $a\in\partial\Omega\cap(K\backslash\mathrm{supp}(g))$, we have, by Lemma 3.1, $P(a+t\mathbf{n}(a))=a$ for t small enough, so that $g(P(a+t\mathbf{n}(a)))=g(a)=0$; this implies $\gamma_\delta(a+t\mathbf{n}(a))=\gamma_\delta(a)=0=g(a)$.
- If $a\in\partial\Omega\cap\mathrm{supp}(g)$, then, for t small enough, $\theta(a+t\mathbf{n}(a))=\theta(a)=1$ ($\theta\equiv 1$ on a neighborhood of $\mathrm{supp}(g)\cap\partial\Omega$) and $P(a+t\mathbf{n}(a))=a$, so that

$$\gamma_\delta(a+t\mathbf{n}(a))-\gamma_\delta(a)=g(a)h_\delta(\mathbf{n}(a)\cdot(a+t\mathbf{n}(a)-a))-g(a)h_\delta(\mathbf{n}(a)\cdot(a-a))=g(a)h_\delta(t).$$

  Since $h_\delta(0)=0$ and $h'_\delta(0)=1$, we deduce that $\frac{\partial\gamma_\delta}{\partial\mathbf{n}}(a)$ exists and is equal to $g(a)$.

Let us now prove that $\gamma_\delta\to 0$ in $W^{1,p}(\Omega)$ as $\delta\to 0$; this will conclude the proof of the lemma.
Notice first that, for all $x\in\Omega$, $|\gamma_\delta(x)|\leq\delta||h||_{L^\infty(\mathbb{R})}||\theta||_{L^\infty(\mathbb{R}^N)}||g||_{L^\infty(\mathbb{R}^N)}$; thus, as $\delta\to 0$, $\gamma_\delta\to 0$ in $L^\infty(\Omega)$ (and also in $L^p(\Omega)$).
Since h_δ is regular and $\theta g\circ P$, $n\circ P\cdot(Id-P)$ are Lipschitz-continuous, we have, on U,

$$\begin{aligned}\nabla\gamma_\delta &= h_\delta(\mathbf{n}\circ P\cdot(Id-P))\nabla(\theta g\circ P)\\ &\quad+\theta g\circ P h'_\delta(\mathbf{n}\circ P\cdot(Id-P))\nabla(\mathbf{n}\circ P\cdot(Id-P)).\end{aligned}$$

But $||h_\delta(\mathbf{n}\circ P\cdot(Id-P))\nabla(\theta g\circ P)||_{L^\infty(\Omega)}\leq\delta||h||_{L^\infty(\mathbb{R})}||\nabla(\theta g\circ P)||_{L^\infty(\Omega)}\to 0$ as $\delta\to 0$. Moreover, by item iii) of Lemma 3.1, for all $y\in U\cap\Omega$, $\mathbf{n}(P(y))\cdot(y-P(y))\neq 0$, so that $h'_\delta(n(P(y))\cdot(y-P(y)))\to 0$ as

$\delta \to 0$ (the support of h'_δ is included in $]-\delta,\delta[$); thus, $\theta g\circ Ph'_\delta(\mathbf{n}\circ P\cdot(Id-P))\nabla(\mathbf{n}\circ P\cdot(Id-P))\to 0$ on Ω; since $||h'_\delta||_{L^\infty(\mathbb{R})}\le ||h'||_{L^\infty(\mathbb{R})}$, we deduce, by the dominated convergence theorem, that $\theta g\circ Ph'_\delta(\mathbf{n}\circ P\cdot(Id-P))\nabla(\mathbf{n}\circ P\cdot(Id-P))\to 0$ in $L^p(\Omega)$ ($n\circ P\cdot(Id-P)$ is Lipschitz-continuous on U, thus its gradient is essentially bounded on U). ■

Proof of Proposition 3.1
Take U an open set containing $\partial\Omega$ such that the projection $P:U\to\partial\Omega$ is well-defined and $C^{k,1}$-continuous. We also choose $\Theta\in C_c^\infty(\mathrm{int}(K)\cap U)$ such that $\Theta\equiv 1$ on a neighborhood of $\mathrm{supp}(u)\cap\partial\Omega$. We define then, on U, $g(x)=\Theta(x)(\nabla u\cdot\mathbf{n})(P(x))$.
g is well-defined and $C^{k,1}$-continuous on U; since its support is a compact subset of U, extending g by 0 outside U, we can suppose that g is well-defined and $C^{k,1}$-continuous on $\mathbb{R}^N$. We also have $g_{|\partial\Omega}=\nabla u\cdot\mathbf{n}$.

By Lemma 3.2, we can find, for all $n\ge 1$, $\gamma_n\in C^{k,1}(\mathbb{R}^N)$ with support in the interior of K such that $||\gamma_n||_{W^{1,p}(\Omega)}\le 1/n$ and $\frac{\partial\gamma_n}{\partial\mathbf{n}}=g=\nabla u\cdot\mathbf{n}$ on $\partial\Omega$. The sequence $(u-\gamma_n)_{n\ge1}$ satisfies the conclusions of the proposition. ■

4 Two applications and a generalization

4.1 Application to the weak formulation of Neumann problems

The classical variational formulation of the Neumann problem

$$\begin{cases} -\Delta u = L & \text{in } \Omega,\\ \nabla u\cdot\mathbf{n}=0 & \text{on } \partial\Omega,\end{cases}\tag{4.1}$$

is the following:

$$\begin{cases} u\in H^1(\Omega),\\ \displaystyle\int_\Omega \nabla u\cdot\nabla\varphi = \langle L,\varphi\rangle_{(H^1(\Omega))',H^1(\Omega)}\,,\ \forall\varphi\in H^1(\Omega).\end{cases}\tag{4.2}$$

With Theorem 1.2 or 1.1 and an integrate by parts, we see that (4.2) is equivalent, if Ω has a $C^{k+1,1}$-continuous boundary (with $k\in\mathbb{N}\backslash\{0\}$ or $k=\infty$) or is polygonal (in which case we take $k=\infty$), to

$$\begin{cases} u\in H^1(\Omega),\\ \displaystyle -\int_\Omega u\Delta\varphi = \langle L,\varphi\rangle_{(H^1(\Omega))',H^1(\Omega)}\,,\ \forall\varphi\in C^{k,1}(\mathbb{R}^N) \text{ such that } \nabla\varphi\cdot\mathbf{n}=0 \text{ on } \partial\Omega.\end{cases}\tag{4.3}$$

Theorems 1.2 and 1.1 allow thus, exactly as for the Dirichlet problem, to write, in certain cases, a formulation of (4.1) — equivalent to the variational formulation, thus leading to existence and uniqueness of a solution — in which all the derivatives appear on the test functions.
This formulation can be useful, for example, to simplify the convergence proof of the finite volume discretization of the Neumann problem on polygonal open sets (see [5]): (4.3) allows to prove that the finite volume approximation converges to the variational solution without using a discrete trace theorem, with the same methods as in the Dirichlet case (notice however that a discrete trace theorem is needed to obtain estimates on the solution of the dicretized Neumann problem).

With Theorem 4.1 below, we can do the same for mixed Dirichlet-Neumann boundary problems.

4.2 Application to the convergence of a finite volume scheme

In [8], the authors prove the convergence of a finite volume scheme for a diffusion problem with mixed Dirichlet-Neumann-Signorini boundary conditions. It is classical, when studying finite volume schemes,

to consider polygonal open sets of $\mathbb{R}^N$ (see [5]); in [8], the authors must however make an additional hypothesis on the open set: they must assume that it is convex.
This restriction comes from the same restriction as in item iii) of Remark 1.2: the authors need to make sure that an element of a Hodge decomposition is in H^2, thus the assumption on the convexity of the open set (this element comes from the resolution of a Neumann problem).

Theorem 1.1 allows to see that the results of [8] are still true without the convexity hypothesis on the open set; moreover, it also simplifies quite a lot the proof of the result in [8] in which the Hodge decomposition is involved (with Theorem 1.1, the functions appearing in this proof are not only in H^2, but also C^∞-continuous, and the error estimates are thus much easier to obtain).

We shall outline another application of our results to finite volume scheme in item ii) of Remark 4.1.

4.3 Generalization to mixed Dirichlet-Neumann boundary conditions

We now consider the case of mixed homogeneous Dirichlet-Neumann boundary conditions. We take thus a measurable subset Γ of $\partial\Omega$ and we consider regular functions which vanish on Γ and whose normal derivatives vanish on $\partial\Omega\backslash\Gamma$.
The trace operator on $\partial\Omega$ being continuous on $W^{1,p}(\Omega)$, if a function u is a limit in $W^{1,p}(\Omega)$ of regular functions which vanish on Γ, u must also vanish on Γ. We denote by $W^{1,p}_\Gamma(\Omega)$ the subset of $W^{1,p}(\Omega)$ made of the functions which vanish σ-a.e. on Γ.

We can now wonder if a space of regular functions which vanish on Γ and whose normal derivatives vanish on $\partial\Omega\backslash\Gamma$ is dense in $W^{1,p}_\Gamma(\Omega)$.
The answer to this question is, for a general Γ, no. Indeed, there exists (see [4], subsection 2.2.1) regular open sets Ω such that, for some measurable $\Gamma\subset\partial\Omega$,

- there exists $u\in W^{1,2}_\Gamma(\Omega)$ such that u does not vanish σ-a.e. on $\partial\Omega$,
- any continuous function on $\partial\Omega$ which vanishes on Γ also vanishes on $\partial\Omega$.

For such Ω and Γ, a space of continuous functions which vanish on Γ cannot be dense in $W^{1,2}_\Gamma(\Omega)$.
We have thus, in order to find a generalization of Theorems 1.1 and 1.2 to mixed Dirichlet-Neumann boundary conditions, to impose some hypotheses on Γ.

The assumption we make on Γ is that Γ and $\partial\Omega\backslash\Gamma$ are "well separated" (the counter-example of [4] is based on a Γ which is a dense open subset of $\partial\Omega$ of very small σ-measure).
The precise formulation of this hypothesis (H) is the following: by denoting

$$B_+=\{(y',y_N)\in B_N(1)\mid y_N>0\}\,,\ D=\{(y',0)\in B_N(1)\}\,,$$
$$B_{++}=\{(y'',y_{N-1},y_N)\in B_+\mid y_{N-1}>0\}\ \text{ and }\ D_+=\{(y'',y_{N-1},0)\in D\mid y_{N-1}\geq 0\}$$

(see figures 1 and 2), we assume that Γ is closed and that, for all $a\in\Gamma$, there exists an open U of $\mathbb{R}^N$ containing a and a Lipschitz-continuous homeomorphism $\phi: U\to B_N(1)$ with a Lipschitz-continuous inverse mapping such that one of the following situation happens:

i) $U\cap\Gamma=U\cap\partial\Omega$, $\phi(U\cap\Omega)=B_+$ and $\phi(U\cap\partial\Omega)=D$: in U, we only see the Dirichlet condition and the geometry is equivalent to that of a half-ball (see figure 1),

ii) $\phi(U\cap\Omega)=B_{++}$, $\phi(U\cap\partial\Omega)=D_+\cup\{(y'',0,y_N)\in B_N(1)\mid y_N>0\}$ and $\phi(U\cap\Gamma)=D_+$: in U, we see both the Neumann and the Dirichlet boundary conditions, but the geometry is equivalent to that of one-fourth of a ball, with the Neumann and Dirichlet conditions on orthogonal hyperplanes (see figure 2).

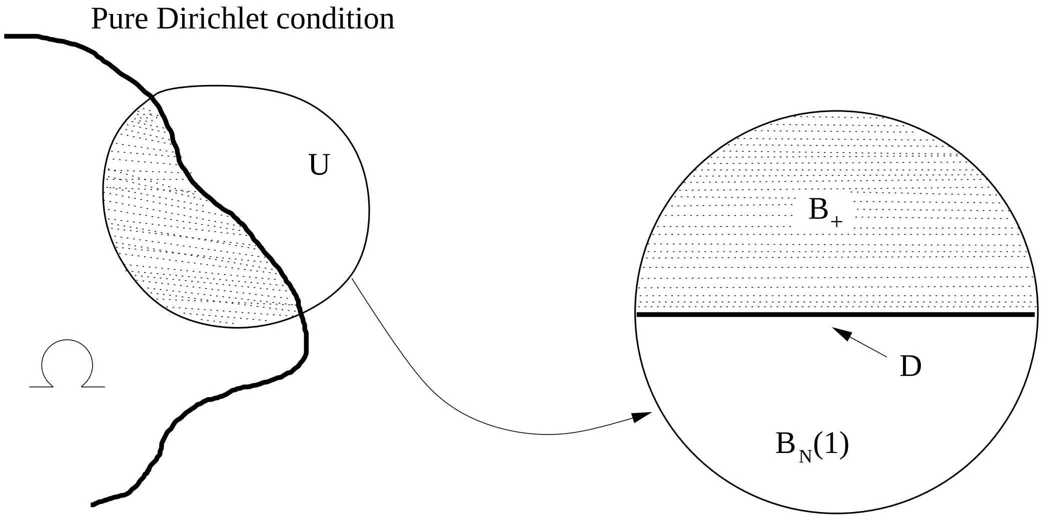

Figure 1 : mapping in the case of non-mixed boundary conditions

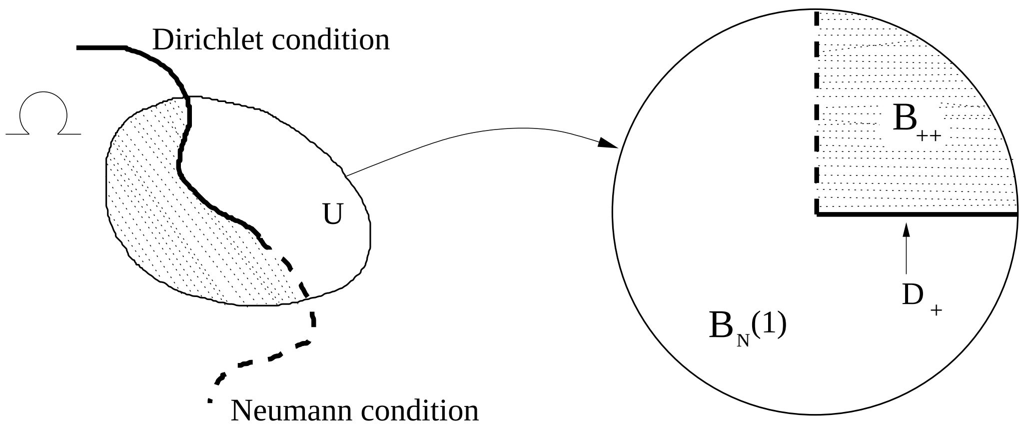

Figure 2 : mapping in the case of mixed boundary conditions

A fundamental example of such a situation is the following: $\Omega = O\times]0,T[$, where O is an open set of $\mathbb{R}^{N-1}$ with a Lipschitz-continuous boundary, and $\Gamma = \overline{O} \times \{T\}$.

We can now state the generalization of Theorems 1.1 and 1.2.

Theorem 4.1 *Let $p \in [1,+\infty[$. We suppose that Ω is an open set of $\mathbb{R}^N$ with a $C^{k+1,1}$-continuous boundary (for a $k \in \mathbb{N} \cup \{\infty\}$) or that Ω is a polygonal open set of $\mathbb{R}^N$ (in which case we let $k = \infty$). If $\Gamma \subset \partial\Omega$ satisfies hypothesis (H), then*

$$\left\{\varphi_{|\Omega} \ : \ \varphi \in C^{k,1}(\mathbb{R}^N)\,,\ \varphi = 0 \text{ on a neighborhood of } \Gamma\,,\ \frac{\partial\varphi}{\partial\mathbf{n}} = 0 \ \sigma\text{-a.e. on } \partial\Omega\right\}$$

is dense in $W^{1,p}_\Gamma(\Omega)$.

Remark 4.1 *i) The same kind of results are true for (Ω,Γ) that can be locally transformed, by a diffeomorphism preserving the outer normal, into $(\widetilde{\Omega},\widetilde{\Gamma})$ satisfying the hypotheses of Theorem 4.1 (see Remark 2.2). For example, if $\Omega = O\times]0,T[$, where O is an open set of $\mathbb{R}^{N-1}$ with a $C^{k+1,1}$-continuous boundary ($k \geq 1$), and $\Gamma = \overline{O} \times \{T\}$, the result of Theorem 4.1 holds.*

ii) In [9], the author uses a similar result to prove the convergence of a finite volume scheme for a diffusion and non-instantaneous dissolution problem in porous medium, when the medium is represented by an open set with regular boundary (at least $C^{2,1}$-continuous, see item i) of Remark 1.2). Theorem 4.1 allows to extend the results of [9] to polygonal open sets, which are the most natural when dealing with finite volume schemes.

Proof of Theorem 4.1

Step 1: we prove that any function $u \in W^{1,p}_\Gamma(\Omega)$ can be approximated in $W^{1,p}(\Omega)$ by functions in $C^\infty_c(\mathbb{R}^N)$ whose supports do not intersect Γ.

We cover the compact set Γ by a finite number of mappings $(U_i,\phi_i)_{i\in[1,l]}$ given by hypothesis (H). We take, for $i \in [1,l]$, $\theta_i \in C^\infty_c(U_i)$ such that $\sum_{i=1}^l \theta_i \equiv 1$ on the neighborhood of Γ.
Since $u = \sum_{i=1}^l \theta_i u + (1 - \sum_{i=1}^l \theta_i)u$, it is sufficient, to conclude this step, to approximate each $\theta_i u$ ($i = 1,\dots,l$) and $(1 - \sum_{i=1}^l \theta_i)u$ by regular functions whose supports do not intersect Γ.

Let us first handle the case of $(1 - \sum_{i=1}^l \theta_i)u$. We take $w_n \in C^\infty_c(\mathbb{R}^N)$ which converges to u in $W^{1,p}(\Omega)$. Then, $(1 - \sum_{i=1}^l \theta_i)w_n \to (1 - \sum_{i=1}^l \theta_i)u$ in $W^{1,p}(\Omega)$, $(1 - \sum_{i=1}^l \theta_i)w_n \in C^\infty_c(\mathbb{R}^N)$ and the support of $(1 - \sum_{i=1}^l \theta_i)w_n$ does not intersect Γ, since $\sum_{i=1}^l \theta_i = 1$ on a neighborhood of Γ. Thus, $(1 - \sum_{i=1}^l \theta_i)u$ can be approximated by regular functions whose supports do not intersect Γ.

Let us now prove that the same result holds true for $u_i = \theta_i u$ ($i = 1,\dots,l$). The function $v_i = u_i \circ \phi_i^{-1}$ is in $W^{1,p}_{\phi_i(U_i\cap\Gamma)}(\phi_i(U_i \cap \Omega))$ and its support is relatively compact in $B_N(1)$ (it is included in the support of $\theta_i \circ \phi_i^{-1}$). We will now handle separately the cases i) or ii) in hypothesis (H).

- Case i): in this case, $\phi_i(U_i \cap \Omega) = B_+$, $\phi_i(U_i \cap \Gamma) = D$ and $v_i \in W^{1,p}_D(B_+)$. v_i vanishing on D, we can extend it to $B_N(1)$ by 0 outside B_+: this gives a function $V_i \in W^{1,p}(B_N(1))$ with compact support in $B_N(1)$ and which vanishes on $B_N(1)\backslash B_+$. A small translation of V_i in direction $(0,\dots,0,1)$ creates a function of $W^{1,p}(B_N(1))$, with compact support in $B_N(1)$, which vanishes on a neighborhood of D and which is as close to V_i (in $W^{1,p}(B_N(1))$) as we want.
- Case ii): in this case, $\phi_i(U_i\cap\Omega) = B_{++}$, $\phi_i(U_i\cap\Gamma) = D_+$ and $v_i \in W^{1,p}_{D_+}(B_{++})$. We extend v_i to B_+ by making an even reflection with respect to the hyperplane $y_{N-1} = 0$; this gives $W_i \in W^{1,p}_D(B_+)$. Then, as in case i), we can find a function in $W^{1,p}(B_N(1))$, with compact support in $B_N(1)$, vanishing on a neighborhood of D and which is as close to W_i (in $W^{1,p}(B_+)$) as we want.

In both cases, we see that v_i can be approximated by restrictions to $\phi_i(U_i \cap \Omega)$ of elements in $W^{1,p}(B_N(1))$ which have compact supports in $B_N(1)$ and vanish on neighborhoods of $\phi_i(U_i \cap \Gamma)$. Transporting this result by ϕ_i, we deduce that we can approximate u_i by restrictions to $U_i \cap \Omega$ of functions in $W^{1,p}(U_i)$ which have compact supports in U_i and vanish on neighborhoods of $U_i \cap \Gamma$.
Since these approximating functions have compact supports in U_i, we can extend them by 0 outside U_i. The convolution of the resulting functions with smoothing kernels give regular functions which approximate u_i and have compact supports that do not intersect Γ.

Step 2: To prove the theorem, it is thus sufficient to approximate, in $W^{1,p}(\Omega)$, any function $u \in C_c^\infty(\mathbb{R}^N)$ whose support does not intersect Γ.

Let K be a compact subset of $\mathbb{R}^N$ containing supp(u) in its interior and such that $K \cap \Gamma = \emptyset$.
If Ω has a $C^{k+1,1}$-continuous boundary (for a $k \geq 0$), Proposition 3.1 concludes the proof.
If Ω is a polygonal open set, Lemma 2.1 allows to see, by induction, that there exists a sequence of functions $u_n \in C_c^\infty(\mathrm{int}(K))$ satisfying $\mathcal{B}_{N-1}$ and converging to u in $W^{1,p}(\Omega)$. As we have seen in the proof of Theorem 1.1, regular functions satisfying $\mathcal{B}_{N-1}$ also satisfy the homogeneous Neumann boundary condition on $\partial\Omega$, and the proof is thus completed. ■

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