Fractal first order partial differential equations

Jérôme Droniou*, Cyril Imbert*†
29/09/2005

Abstract

The present paper is concerned with semilinear partial differential equations involving a particular pseudo-differential operator. It investigates both fractal conservation laws and non-local Hamilton-Jacobi equations. The idea is to combine an integral representation of the operator and Duhamel's formula to prove, on the one side, the key *a priori* estimates for the scalar conservation law *and* the Hamilton-Jacobi equation and, on the other side, the smoothing effect of the operator. As far as Hamilton-Jacobi equations are concerned, a non-local vanishing viscosity method is used to construct a (viscosity) solution when existence of regular solutions fails, and a rate of convergence is provided. Turning to conservation laws, global-in-time existence and uniqueness are established. We also show that our formula allows to obtain entropy inequalities for the non-local conservation law, and thus to prove the convergence of the solution, as the non-local term vanishes, toward the entropy solution of the pure conservation law.

Mathematical subject classifications: 35B45, 35B65, 35A35, 35S30
Keywords: Lévy operator, fractal conservation laws, maximum principle, non-local Hamilton-Jacobi equations, smoothing effect, *a priori* estimates, global-in-time existence, rate of convergence.

1 Introduction

In this paper, we are interested in solving semilinear partial differential equations involving the fractal operator, also called Lévy operator, defined on the Schwartz class $S(\mathbb{R}^N)$ by

$$g_\lambda[\varphi] = \mathcal{F}^{-1}\left(|\cdot|^\lambda \mathcal{F}(\varphi)\right) \qquad \text{with } 0 < \lambda < 2 \tag{1.1}$$

where $\mathcal{F}$ is the Fourier transform. The study of PDEs involving g_λ is motivated by a number of physical problems, such as overdriven detonations in gases [9] or anomalous diffusion in semiconductor growth [27], and by mathematical models in finance (see below for references). We consider perturbations by g_λ of Hamilton-Jacobi equations or scalar conservation laws, that is to say

$$\begin{cases} \partial_t u(t,x) + g_\lambda[u(t,\cdot)](x) = F(t,x,u(t,x),\nabla u(t,x)) & t>0\,,\ x\in\mathbb{R}^N\,, \\ u(0,x) = u_0(x) & x\in\mathbb{R}^N \end{cases} \tag{1.2}$$

or

$$\begin{cases} \partial_t u(t,x) + \operatorname{div}(f(t,x,u(t,x))) + g_\lambda[u(t,\cdot)](x) = h(t,x,u(t,x)) & t>0\,,\ x\in\mathbb{R}^N\,, \\ u(0,x) = u_0(x) & x\in\mathbb{R}^N. \end{cases} \tag{1.3}$$

These kinds of equations have already been studied. As far as scalar conservation laws are concerned, some recent papers have investigated them. One of the first works on this subject is probably [4], which deals with (1.3) when $h=0$ and $f(t,x,u)=f(u)$; using energy estimates, it states some local-in-time existence and uniqueness results of weak solutions if f has a polynomial growth. This result is strengthened in [15], where a splitting method is used to prove global-in-time existence and uniqueness of a regular solution if $\lambda > 1$.

To our best knowledge, Hamilton-Jacobi equations of type (1.2) first appeared in the context of mathematical finance as Bellman equations of optimal control of jump diffusion processes [24]. See also [25, 23, 1] and more recently [6, 7, 8]. A general theory for non-linear integro-partial differential equations is developed by Jakobsen and Karlsen [19, 20]. Some of the ideas of [15] are adapted in [18] to prove that

*Département de Mathématiques, UMR CNRS 5149, CC 051, Université Montpellier II, Place Eugène Bataillon, 34095 Montpellier cedex 5, France. E-mails: droniou@math.univ-montp2.fr, imbert@math.univ-montp2.fr
†Polytech'Montpellier, Université Montpellier II, Montpellier, France

(1.2) has regular solutions if $\lambda > 1$. It adapts previous viscosity solution theories to the equation (1.2) (existence via Perron's method, comparison results, stability) and use techniques of [15] to obtain further regularity.

The preceding methods to handle (1.2) and (1.3) are somewhat incompatible: the splitting method for Hamilton-Jacobi equation (1.2) is less direct than Perron's one and it is well known that the notion of viscosity solution is inadequate to conservation laws such as (1.3). However, a scalar conservation law can always be formally written as a Hamilton-Jacobi equation (write $\mathrm{div}\,(f(u)) = f'(u) \cdot \nabla u$). In this paper, we present a way to simultaneously solve (1.2) and (1.3) by using this simple fact and the construction of regular solutions to the Hamilton-Jacobi equations under weak assumptions (that are satisfied by both (1.2) and (1.3)). The key estimate is given by Proposition 3.1. As we notice in the course of the proofs, the method we use is also valid for more general operators than g_λ.

The starting point of this work is the use of an equivalent definition of the fractal operator, namely an integral formula for g_λ similar to the ones appearing in [11] and [18]. This formula permits to extend the operator from Schwartz functions to C^2_b ones and is moreover used to establish what we call a "reverse maximum principle" that says, freely speaking, that $g_\lambda[\phi](x)$ is nonnegative if x is a maximum point of ϕ. This principle is the key point when proving the estimates for the regular solutions of (1.2). Thanks to the Fourier definition of g_λ, we are also able to give properties of the kernel associated to g_λ and thus to write a Duhamel formula for the solutions of the PDEs.
The main novelty of this paper is to combine the "reverse maximum principle" (coming from the integral formula for g_λ) and the Duhamel formula in order to prove existence and uniqueness of global smooth solutions to our PDEs.
As far as Hamilton-Jacobi equations are concerned, the study of (1.2) in [18] is made for $\lambda > 1$ and by using Perron's method. We generalize here the results of this paper to the case $\lambda \in]0, 2[$ and we weaken the hypotheses on the Hamiltonian. We first prove that the fractal operator has a smoothing effect for $\lambda \in]1, 2[$ under very general (and natural) hypotheses on F; the idea to obtain a global solution is, roughly speaking, to study how $\sup_{\mathbb{R}^N} |u(t, \cdot)|$ evolves. We next treat the case $\lambda \in]0, 2[$ (recall that, contrary to the formula in [18], ours is valid for such λ) by solving (1.2) in the sense of viscosity solutions; as expected in this context (see (1.1)), these solutions are no more regular but only bounded and uniformly continuous. We use a non-local vanishing viscosity method (though we could have used Perron's method, see Remark 3.6): precisely, we add a vanishing fractal operator εg_μ with $\mu > 1$ and we pass to the limit $\varepsilon \to 0$. We also provide, for all $\mu \in]0, 2[$, a rate of convergence that is in some respect surprising, compared with the case $\mu > 1$ treated in [18]. Let us also mention that the reverse maximum principle and its main consequence, namely the key estimate given by Proposition 3.1, can be generalized to the framework of viscosity solutions: nonsmooth versions of both results are stated and proved in Appendix. We have chosen not to use these versions because, anywhere we can, we search for regular solutions and we turn to the notion of viscosity solution only if mandatory (that is if $\lambda \leq 1$; see Subsection 3.2).
The case of scalar hyperbolic equations (1.3) with $\lambda > 1$ is treated next. The *a priori* estimates on the solution follow from Proposition 3.1, the same proposition that gives the *a priori* estimates for (1.2), and the existence of a regular solution is as straightforward. The splitting method of [15] can be adapted to some cases where f and h depend on (t, x) (see [14]), but this is awfully technical; the technique we use here therefore presents a noticeable simplification in the study of (1.3) in the general case. The question of non-local vanishing viscosity regularization (multiplying $g_\lambda[u]$ by ε and letting $\varepsilon \to 0$) is treated in [13], still using the splitting method (and for $h = 0$, $f(t, x, u) = f(u)$); at the end of the present paper, we quickly indicate how the formula for g_λ allows to significantly simplify the corresponding proofs for general f and h, in particular the proof of the entropy inequalities for (1.3).

The paper is organised as follows. Section 2 is devoted to give an analytical proof of the integral formula for g_λ (Theorem 2.1). It also contains the "reverse maximum principle" (Theorem 2.2) we presented above. In Section 3, we study Hamilton-Jacobi equations. We first present the smoothing effect of the fractal operator on (1.2) for $\lambda \in]1, 2[$ (Theorem 3.1); next, we construct viscosity solutions for $\lambda \in]0, 2[$ by a non-local vanishing viscosity method (Theorem 3.3) and we prove a rate of convergence (Theorem 3.4). Section 4 contains the resolution of (1.3) and the (short) proof of the entropy inequalities associated with the perturbed conservation law (Subsection 4.2). Some appendixes in Section 5 conclude the paper. In particular, the reader can find there a generalization of the "reverse maximum principle" and of the key estimate to the viscosity framework.

Notations. Throughout the paper, B_r (resp. $B_r(x)$) denotes the ball of $\mathbb{R}^N$ centered at the origin (resp. at x) and of radius r. Euler's function is denoted by Γ.

2 Integral representation of g_λ

The main result of this section is the integral representation of g_λ, which generalizes Lemma 1 in [18]. As a consequence of this formula, we extend the definition of g_λ from Schwartz functions to C_b^2 functions and we prove what we call a "reverse maximum principle": roughly speaking, it says that at a maximum point of a C_b^2 function φ, we have $g_\lambda[\varphi] \geq 0$. This result is the crucial argument when proving the key estimate stated in Proposition 3.1.

Theorem 2.1 *If $\lambda \in]0,2[$, then, for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$, all $x \in \mathbb{R}^N$ and all $r > 0$,*

$$g_\lambda[\varphi](x) = -c_N(\lambda)\left(\int_{B_r} \frac{\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z}{|z|^{N+\lambda}}\,dz + \int_{\mathbb{R}^N\setminus B_r} \frac{\varphi(x+z)-\varphi(x)}{|z|^{N+\lambda}}\,dz\right) \tag{2.1}$$

where $c_N(\lambda) = \frac{\lambda\Gamma(\frac{N+\lambda}{2})}{2\pi^{\frac{N}{2}+\lambda}\Gamma(1-\frac{\lambda}{2})}$. We can generalize this formula in two cases:

i) If $\lambda \in]0,1[$, we can take $r = 0$:

$$g_\lambda[\varphi](x) = -c_N(\lambda)\int_{\mathbb{R}^N} \frac{\varphi(x+z)-\varphi(x)}{|z|^{N+\lambda}}\,dz.$$

ii) If $\lambda \in]1,2[$, we can take $r = +\infty$:

$$g_\lambda[\varphi](x) = -c_N(\lambda)\int_{\mathbb{R}^N} \frac{\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z}{|z|^{N+\lambda}}\,dz.$$

Before proving these formulae, let us state some of their consequences. We first notice that (2.1) allows to define $g_\lambda[\varphi] \in C_b(\mathbb{R}^N)$ for $\varphi \in C_b^2(\mathbb{R}^N)$. In fact, this gives a continuous extension of g_λ in the following sense.

Proposition 2.1 *Let $\lambda \in]0,2[$ and $\varphi \in C_b^2(\mathbb{R}^N)$. If $(\varphi_n)_{n\geq 1} \in C_b^2(\mathbb{R}^N)$ is bounded in $L^\infty(\mathbb{R}^N)$ and $D^2\varphi_n \to D^2\varphi$ locally uniformly on $\mathbb{R}^N$, then $g_\lambda[\varphi_n] \to g_\lambda[\varphi]$ locally uniformly on $\mathbb{R}^N$.*

Remark 2.1 *We could also define g_λ on Hölder spaces of functions (depending on λ), and state an equivalent of Proposition 2.1 in this framework.*

Proof of Proposition 2.1
The operator g_λ being linear, we can assume that $\varphi = 0$. Since $(\varphi_n)_{n\geq 1}$ is bounded in $L^\infty(\mathbb{R}^N)$, the second integral term of (2.1) applied to $\varphi = \varphi_n$ is small, uniformly for $n \geq 1$ and $x \in \mathbb{R}^N$, if r is large. By Taylor's formula, for $|z| \leq r$ and $|x| \leq R$ we have $|\varphi_n(x+z)-\varphi_n(x)-\nabla\varphi_n(x)\cdot z| \leq ||D^2\varphi_n||_{L^\infty(B_{r+R})}|z|^2$; hence, with r fixed, the first integral term of (2.1) applied to $\varphi = \varphi_n$ is small, uniformly for $x \in B_R$, if n is large. ■

From (2.1) it is obvious that, if x is a global maximum of φ, then $g_\lambda[\varphi](x) \geq 0$, with equality if and only if φ is constant (notice that $c_N(\lambda) > 0$ for $\lambda \in]0,2[$). We have a generalization of this property, which will be the key argument to study first order perturbations of $\partial_t + g_\lambda$.

Theorem 2.2 *Let $\lambda \in]0,2[$ and $\varphi \in C_b^2(\mathbb{R}^N)$. If $(x_n)_{n\geq 1}$ is a sequence of $\mathbb{R}^N$ such that $\varphi(x_n) \to \sup_{\mathbb{R}^N} \varphi$ as $n \to \infty$, then $\lim_{n\to\infty} \nabla\varphi(x_n) = 0$ and $\liminf_{n\to\infty} g_\lambda[\varphi](x_n) \geq 0$.*

Proof of Theorem 2.2
Since the second derivative of φ is bounded, there exists C such that, for all $n \geq 1$ and all $z \in \mathbb{R}^N$,

$$\sup_{\mathbb{R}^N} \varphi \geq \varphi(x_n + z) \geq \varphi(x_n) + \nabla\varphi(x_n) \cdot z - C|z|^2. \tag{2.2}$$

Up to a subsequence, we can assume that $\nabla\varphi(x_n) \to p$ (this sequence is bounded). Passing to the limit $n \to \infty$ in (2.2) gives $0 \geq p \cdot z - C|z|^2$; choosing then $z = tp$ and letting $t \to 0^+$ shows that $p = 0$, which proves that $\lim_{n\to\infty} \nabla\varphi(x_n) = 0$ (the only adherence value of the bounded sequence $(\nabla\varphi(x_n))_{n\geq 1}$ is 0). Since $\varphi(x_n + z) - \varphi(x_n) \leq \sup_{\mathbb{R}^N} \varphi - \varphi(x_n) \to 0$, we deduce that, for all $z \in \mathbb{R}^N$,

$$\begin{array}{l} \limsup_{n\to\infty} (\varphi(x_n + z) - \varphi(x_n)) \leq 0 \\ \limsup_{n\to\infty} (\varphi(x_n + z) - \varphi(x_n) - \nabla\varphi(x_n) \cdot z) \leq 0. \end{array} \tag{2.3}$$

We also have

$$\frac{|\varphi(x_n + z) - \varphi(x_n)|}{|z|^{N+\lambda}} \leq \frac{2||\varphi||_{L^\infty(\mathbb{R}^N)}}{|z|^{N+\lambda}} \in L^1(\mathbb{R}^N \backslash B_r)$$

and

$$\frac{|\varphi(x_n + z) - \varphi(x_n) - \nabla\varphi(x_n) \cdot z|}{|z|^{N+\lambda}} \leq \frac{||D^2\varphi||_{L^\infty(\mathbb{R}^N)}|z|^2}{|z|^{N+\lambda}} \in L^1(B_r).$$

Hence, by (2.3) and Fatou's Lemma,

$$0 \geq \int_{\mathbb{R}^N \backslash B_r} \limsup_{n\to\infty} \frac{\varphi(x_n + z) - \varphi(x_n)}{|z|^{N+\lambda}} \, dz \geq \limsup_{n\to\infty} \int_{\mathbb{R}^N \backslash B_r} \frac{\varphi(x_n + z) - \varphi(x_n)}{|z|^{N+\lambda}} \, dz$$

and

$$0 \geq \int_{B_r} \limsup_{n\to\infty} \frac{\varphi(x_n + z) - \varphi(x_n) - \nabla\varphi(x_n) \cdot z}{|z|^{N+\lambda}} \, dz \geq \limsup_{n\to\infty} \int_{B_r} \frac{\varphi(x_n + z) - \varphi(x_n) - \nabla\varphi(x_n) \cdot z}{|z|^{N+\lambda}} \, dz.$$

Combining these inequalities and (2.1) permits to achieve the proof of the theorem. ■

Remark 2.2 *This theorem is also true for $\lambda = 2$, that is to say $g_2 = -4\pi^2\Delta$, provided that $\varphi \in C_b^3(\mathbb{R}^N)$.*

We now conclude this section by proving the formula given for g_λ.

Proof of Theorem 2.1
Step 1: a preliminary formula.
We first assume that $\lambda \in]1,2[$. We have $g_\lambda[\varphi] = \mathcal{F}^{-1}(|\cdot|^\lambda \mathcal{F}(\varphi))$; but $\mathcal{F}(\Delta\varphi) = -4\pi^2|\cdot|^2\mathcal{F}(\varphi)$ and therefore, for $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$g_\lambda[\varphi] = \frac{1}{-4\pi^2}\mathcal{F}^{-1}(|\cdot|^{\lambda-2}\mathcal{F}(\Delta\varphi)). \tag{2.4}$$

Since $\lambda \in]1,2[$, we have $\lambda - 2 \in]-N, 0[$; hence $|\cdot|^{\lambda-2}$ is locally integrable and is in $\mathcal{S}'(\mathbb{R}^N)$. The inverse Fourier transform of $|\cdot|^{\lambda-2}$ is a distribution with radial symmetry and homogeneity of order $-N-(\lambda-2)$; we deduce that there exists $C_N(\lambda)$ such that

$$\mathcal{F}^{-1}(|\cdot|^{\lambda-2}) = C_N(\lambda)|\cdot|^{-N-(\lambda-2)} \tag{2.5}$$

in $\mathcal{D}'(\mathbb{R}^N\backslash\{0\})$; since $|\cdot|^{-N-(\lambda-2)}$ is locally integrable, it is quite easy to see that (2.5) also holds in $\mathcal{S}'(\mathbb{R}^N)$. We compute $C_N(\lambda)$ by taking the test function $\gamma(x) = e^{-\pi|x|^2}$, which is its own inverse Fourier transform:

$$\int_{\mathbb{R}^N} |x|^{\lambda-2}e^{-\pi|x|^2} \, dx = \langle\mathcal{F}^{-1}(|\cdot|^{\lambda-2}), \gamma\rangle_{\mathcal{S}'(\mathbb{R}^N),\mathcal{S}(\mathbb{R}^N)} = C_N(\lambda) \int_{\mathbb{R}^N} |x|^{-N-(\lambda-2)}e^{-\pi|x|^2} \, dx.$$

Using polar coordinates, we deduce $\int_0^\infty r^{N+\lambda-4}e^{-\pi r^2}\,rdr = C_N(\lambda)\int_0^\infty r^{-\lambda}e^{-\pi r^2}\,rdr$ and the change of variable $\tau=\pi r^2$ implies

$$\pi^{-(N+\lambda-4)/2}\int_0^\infty \tau^{(N+\lambda-4)/2}e^{-\tau}\,\frac{d\tau}{2\pi} = C_N(\lambda)\pi^{\lambda/2}\int_0^\infty \tau^{-\lambda/2}e^{-\tau}\,\frac{d\tau}{2\pi}\,,$$

that is to say $C_N(\lambda)=\Gamma(\frac{N+\lambda}{2}-1)/[\pi^{\frac{N}{2}+\lambda-2}\Gamma(1-\frac{\lambda}{2})]$. With this value of $C_N(\lambda)$, (2.4) and (2.5) give, for all $\varphi\in\mathcal{S}(\mathbb{R}^N)$,

$$g_\lambda[\varphi] = -\frac{\Gamma\left(\frac{N+\lambda}{2}-1\right)}{4\pi^{\frac{N}{2}+\lambda}\Gamma\left(1-\frac{\lambda}{2}\right)}|\cdot|^{-N-(\lambda-2)}*\Delta\varphi. \tag{2.6}$$

Step 2: proof of (2.1) for $\lambda\in]1,2[$.
Let $r>0$, $\varphi\in\mathcal{S}(\mathbb{R}^N)$, $x\in\mathbb{R}^N$ and define $\phi_x(z)=\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z\,\theta(z)$, where $\theta\in C_c^\infty(\mathbb{R}^N)$ is even and equal to 1 on B_r. We have $\Delta\phi_x(z)=\Delta\varphi(x+z)-\nabla\varphi(x)\cdot\Delta(z\,\theta(z))$, and thus, with $\beta=-N-(\lambda-2)\in]-N,0[$,

$$|\cdot|^{-N-(\lambda-2)}*\Delta\varphi(x)=\int_{\mathbb{R}^N}|z|^\beta\Delta\varphi(x+z)\,dz=\int_{\mathbb{R}^N}|z|^\beta\Delta\phi_x(z)\,dz+\nabla\varphi(x)\cdot\int_{\mathbb{R}^N}|z|^\beta\Delta(z\,\theta(z))\,dz$$

(all these functions are integrable since $\Delta\varphi(x+z)$ and $\Delta(z\,\theta(z))$ are both Schwartz functions). But $z\mapsto z\,\theta(z)$ is odd, so $z\mapsto|z|^\beta\Delta(z\,\theta(z))$ is also odd and its integral on $\mathbb{R}^N$ vanishes. Hence,

$$|\cdot|^{-N-(\lambda-2)}*\Delta\varphi(x)=\int_{\mathbb{R}^N}|z|^\beta\Delta\phi_x(z)\,dz=\lim_{\varepsilon\to0}\int_{C_\varepsilon}|z|^\beta\Delta\phi_x(z)\,dz \tag{2.7}$$

where $C_\varepsilon=\{\varepsilon\le|z|\le1/\varepsilon\}$. By Green's formula,

$$\int_{C_\varepsilon}|z|^\beta\Delta\phi_x(z)\,dz=\int_{C_\varepsilon}\Delta(|z|^\beta)\phi_x(z)\,dz+\int_{\partial C_\varepsilon}\Big[|z|^\beta\nabla\phi_x(z)\cdot\mathbf{n}(z)-\phi_x(z)\nabla(|z|^\beta)\cdot\mathbf{n}(z)\Big]\,d\sigma_\varepsilon(z) \tag{2.8}$$

where σ_ε is the $(N-1)$-dimensional measure on $\partial C_\varepsilon=S_\varepsilon\cup S_{1/\varepsilon}$ (with $S_a=\{|z|=a\}$) and $\mathbf{n}$ is the outer unit normal to C_ε. On a neighbourhood of 0, we have $\phi_x(z)=\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z$, and thus $\phi_x(z)=\mathcal{O}(|z|^2)$, $\nabla\phi_x(z)=\mathcal{O}(|z|)$; using $|\nabla(|z|^\beta)|=|\beta|\,|z|^{\beta-1}$, we deduce, since $N+\beta=2-\lambda>0$,

$$\left|\int_{S_\varepsilon}\big(|z|^\beta\nabla\phi_x(z)\cdot\mathbf{n}(z)-\phi_x(z)\nabla(|z|^\beta)\cdot\mathbf{n}(z)\big)\,d\sigma_\varepsilon(z)\right|\le C\varepsilon^{N-1}\varepsilon^{\beta+1}\to0\qquad\text{as }\varepsilon\to0. \tag{2.9}$$

Since $-(N-1)-(\beta-1)=2-(N+\beta)=\lambda>0$ and, at infinity, $\phi_x(z)=\varphi(x+z)-\varphi(x)$ is bounded and $\nabla\phi_x(z)=\nabla\varphi(x+z)$ is rapidly decreasing, we obtain

$$\begin{aligned}&\left|\int_{S_{1/\varepsilon}}\big(|z|^\beta\nabla\phi_x(z)\cdot\mathbf{n}(z)-\phi_x(z)\nabla(|z|^\beta)\cdot\mathbf{n}(z)\big)\,d\sigma_\varepsilon(z)\right|\\&\quad\le\ C\left(\frac{1}{\varepsilon}\right)^{N-1+\beta}\sup_{S_{1/\varepsilon}}|\nabla\varphi(x+\cdot)|+C\varepsilon^{-(N-1)}\varepsilon^{-(\beta-1)}\to0\qquad\text{as }\varepsilon\to0.\end{aligned} \tag{2.10}$$

An easy computation gives

$$\Delta(|z|^\beta)=\mathrm{div}(\beta|z|^{\beta-2}z)=\beta\left(N|z|^{\beta-2}+(\beta-2)|z|^{\beta-3}\frac{z}{|z|}\cdot z\right)=\beta(N+\beta-2)|z|^{\beta-2}$$

and therefore

$$\int_{C_\varepsilon}\Delta(|z|^\beta)\phi_x(z)\,dz=(N+\lambda-2)\lambda\int_{C_\varepsilon}|z|^{-N-\lambda}\phi_x(z)\,dz. \tag{2.11}$$

Since $\phi_x(z) = \mathcal{O}(|z|^2)$ on a neighbourhood of 0 and ϕ_x is bounded on $\mathbb{R}^N$, the function $|\cdot|^{-N-\lambda}\phi_x$ is integrable on $\mathbb{R}^N$ and we can pass to the limit as $\varepsilon \to 0$ in the right-hand side of (2.11).

Combining (2.7), (2.8), (2.9), (2.10) and (2.11) yields

$$|\cdot|^{-N-(\lambda-2)} * \Delta\varphi(x) = \lambda(N+\lambda-2)\int_{\mathbb{R}^N} \frac{\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z\,\theta(z)}{|z|^{N+\lambda}}\,dz.$$

Since $\theta = 1$ on B_r, this gives

$$\begin{aligned}|\cdot|^{-N-(\lambda-2)} * \Delta\varphi(x) &= \lambda(N+\lambda-2)\int_{B_r} \frac{\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z}{|z|^{N+\lambda}}\,dz \\ &\quad +\lambda(N+\lambda-2)\int_{\mathbb{R}^N\backslash B_r} \frac{\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z\,\theta(z)}{|z|^{N+\lambda}}\,dz.\end{aligned}$$

But $|z|^{-N-\lambda}(\varphi(x+z)-\varphi(x))$ and $|z|^{-N-\lambda}z\,\theta(z)$ are integrable on $\mathbb{R}^N\backslash B_r$, and thus

$$\int_{\mathbb{R}^N\backslash B_r} \frac{\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z\,\theta(z)}{|z|^{N+\lambda}}\,dz = \int_{\mathbb{R}^N\backslash B_r} \frac{\varphi(x+z)-\varphi(x)}{|z|^{N+\lambda}}\,dz - \nabla\varphi(x)\cdot\int_{\mathbb{R}^N\backslash B_r} \frac{z\,\theta(z)}{|z|^{N+\lambda}}\,dz.$$

Since $z \mapsto |z|^{-N-\lambda}z\,\theta(z)$ is odd, this last integral vanishes and we deduce

$$\begin{aligned}|\cdot|^{-N-(\lambda-2)} * \Delta\varphi(x) &= \lambda(N+\lambda-2)\int_{B_r} \frac{\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z}{|z|^{N+\lambda}}\,dz \\ &\quad +\lambda(N+\lambda-2)\int_{\mathbb{R}^N\backslash B_r} \frac{\varphi(x+z)-\varphi(x)}{|z|^{N+\lambda}}\,dz.\end{aligned}$$

Using this formula in (2.6) and taking into account $(N+\lambda-2)\Gamma(\frac{N+\lambda}{2}-1) = 2(\frac{N+\lambda}{2}-1)\Gamma(\frac{N+\lambda}{2}-1) = 2\Gamma(\frac{N+\lambda}{2})$, we obtain (2.1) if $\lambda \in]1,2[$.

Notice that, up to now, the reasoning is also valid for any $\lambda \in]0,2[$ if $N \geq 2$. To prove (2.1) in the general case, we must use a holomorphy argument.

Step 3: conclusion.
We now obtain (2.1) in the case $\lambda \in]0,1]$. Let $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. Since $\mathcal{F}(\varphi) \in \mathcal{S}(\mathbb{R}^N)$, we have, for all λ in the strip $E = \{\lambda \in \mathbb{C} \mid 0 < \mathrm{Re}(\lambda) < 2\}$,

$$|\,|\cdot|^\lambda \mathcal{F}(\varphi)\,| = |\cdot|^{\mathrm{Re}(\lambda)}|\mathcal{F}(\varphi)| \leq (1+|\cdot|^2)|\mathcal{F}(\varphi)| \in L^1(\mathbb{R}^N).$$

Hence, by holomorphy under the integral sign, the function

$$\lambda \mapsto g_\lambda[\varphi](x) = \int_{\mathbb{R}^N} e^{2i\pi x\cdot\xi}|\xi|^\lambda \mathcal{F}(\varphi)(\xi)\,d\xi$$

is holomorphic on E. For all $0 < a \leq \mathrm{Re}(\lambda) \leq b < 2$, the integrands in (2.1) are bounded by integrable functions which only depend on a and b: if $z \in B_r$, we have $|\,|z|^\lambda\,| = |z|^{\mathrm{Re}(\lambda)} \geq r^{\mathrm{Re}(\lambda)-b}|z|^b \geq c_{r,b}|z|^b$ and, if $z \notin B_r$, then $|\,|z|^\lambda| \geq r^{\mathrm{Re}(\lambda)-a}|z|^a \geq c'_{r,a}|z|^a$; hence, the two integral terms in this formula are also holomorphic with respect to $\lambda \in E$. Since $\lambda \mapsto c_N(\lambda)$ is holomorphic on E (Γ is holomorphic in the half-plane $\{\mathrm{Re} > 0\}$ and has no zero), all the functions of λ in (2.1) are holomorphic on E; this equality being satisfied for all real λ in $]1,2[$, it holds in fact for any $\lambda \in E$. In particular, this proves (2.1) if $\lambda \in]0,2[$.

The special cases i) and ii) of the theorem are easy consequences of (2.1). Indeed, $\varphi(x+z)-\varphi(x) = \mathcal{O}(|z|)$ on a neighbourhood of 0; thus, if $\lambda < 1$, $|z|^{-N-\lambda}(\varphi(x+z)-\varphi(x))$ is integrable on $\mathbb{R}^N$ and we can pass to the limit $r \to 0$ in (2.1). The function $z \mapsto \varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z$ has a linear growth at infinity; therefore, if $\lambda > 1$, $|z|^{-N-\lambda}(\varphi(x+z)-\varphi(x)-\nabla\varphi(x)\cdot z)$ is integrable on $\mathbb{R}^N$ and we conclude by letting $r \to \infty$ in (2.1). ■

3 Fractal Hamilton-Jacobi equations

3.1 A smoothing effect for $\lambda \in]1,2[$

We assume here that $\lambda \in]1,2[$ and we consider the Cauchy problem

$$\begin{cases} \partial_t u(t,x) + g_\lambda[u(t,\cdot)](x) = F(t,x,u(t,x),\nabla u(t,x)) & t>0\,,\ x\in\mathbb{R}^N\,, \\ u(0,x) = u_0(x) & x\in\mathbb{R}^N\,, \end{cases} \tag{3.1}$$

where $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ and $F \in C^\infty([0,\infty[\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)$ satisfies

$$\begin{array}{c} \forall T>0\,,\ \forall R>0\,,\ \forall k\in\mathbb{N}\,,\ \exists C_{T,R,k} \text{ such that,} \\ \text{for all } (t,x,s,\xi)\in[0,T]\times\mathbb{R}^N\times[-R,R]\times B_R \text{ and all } \alpha\in\mathbb{N}^{2N+2} \text{ satisfying } |\alpha|\le k, \\ |\partial^\alpha F(t,x,s,\xi)| \le C_{T,R,k}. \end{array} \tag{3.2}$$

We also assume that

$$\begin{array}{c} \forall T>0, \text{ there exists } \Lambda_T : [0,+\infty[\mapsto]0,+\infty[\text{ continuous nondecreasing} \\ \text{such that } \int_0^\infty \frac{1}{\Lambda_T(a)}\,da = +\infty \text{ and, for all } (t,x,s)\in[0,T]\times\mathbb{R}^N\times\mathbb{R}, \\ \mathrm{sgn}(s)F(t,x,s,0) \le \Lambda_T(|s|)\,, \end{array} \tag{3.3}$$

$$\begin{array}{c} \forall T>0, \forall R>0, \text{ there exists } \Gamma_{T,R} : [0,+\infty[\mapsto]0,+\infty[\text{ continuous nondecreasing} \\ \text{such that } \int_0^\infty \frac{1}{\Gamma_{T,R}(a)}\,da = +\infty \text{ and, for all } (t,x,s,\xi)\in[0,T]\times\mathbb{R}^N\times[-R,R]\times\mathbb{R}^N, \\ |\xi|\partial_s F(t,x,s,\xi) \le \Gamma_{T,R}(|\xi|)\,, \quad |\nabla_x F(t,x,s,\xi)| \le \Gamma_{T,R}(|\xi|) \end{array} \tag{3.4}$$

and we define

$$\mathcal{L}_T(a) = \int_0^a \frac{1}{\Lambda_T(b)}\,db \quad \text{and} \quad \mathcal{G}_{T,R}(a) = \int_0^a \frac{1}{2N\Gamma_{T,R}(b)}\,db. \tag{3.5}$$

By the assumptions on Λ_T and $\Gamma_{T,R}$, the functions $\mathcal{L}_T$ and $\mathcal{G}_{T,R}$ are nondecreasing C^1-diffeomorphisms from $[0,\infty[$ to $[0,\infty[$. Our main result concerning (3.1) is the following.

Theorem 3.1 *Let $\lambda \in]1,2[$, $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ and F satisfy (3.2), (3.3) and (3.4). There exists a unique solution u to (3.1) in the following sense: for all $T>0$,*

$$u\in C_b(]0,T[\times\mathbb{R}^N)\,,\ \nabla u\in C_b(]0,T[\times\mathbb{R}^N)^N \ \text{ and, for all } a\in]0,T[\,,\ u\in C_b^\infty(]a,T[\times\mathbb{R}^N)\,, \tag{3.6}$$

$$u \text{ satisfies the PDE of (3.1) on }]0,T[\times\mathbb{R}^N\,, \tag{3.7}$$

$$u(t,\cdot)\to u_0 \text{ uniformly on } \mathbb{R}^N, \text{ as } t\to 0. \tag{3.8}$$

We also have the following estimates on the solution: for all $0<t<T<\infty$,

$$||u(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \le (\mathcal{L}_T)^{-1}\left(t+\mathcal{L}_T(||u_0||_{L^\infty(\mathbb{R}^N)})\right)\,, \tag{3.9}$$

$$||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \le (\mathcal{G}_{T,R})^{-1}\left(t+\mathcal{G}_{T,R}(||Du_0||_{L^\infty(\mathbb{R}^N)})\right)\,, \tag{3.10}$$

where $\mathcal{L}_T$ and $\mathcal{G}_{T,R}$ are defined by (3.5), $||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)} = \sum_{i=1}^N ||\partial_i u(t,\cdot)||_{L^\infty(\mathbb{R}^N)}$ and R is any upper bound of $||u||_{L^\infty(]0,T[\times\mathbb{R}^N)}$.

Remark 3.1 *The uniqueness holds under weaker assumptions (see Corollary 3.1) and, with the technique used in Section 4, it can also be proved if the uniform convergence in (3.8) is replaced by a L^∞ weak-$*$ convergence.*

3.1.1 Discussion of the assumptions

We assume that F is regular because we look here for regular solutions to (3.1) (we relax this in Subsection 3.2); in this framework, (3.2) is restricting only in the sense that it imposes bounds which are uniform with respect to $x \in \mathbb{R}^N$, but this is natural since we want solutions that also satisfy such global bounds.
Assumption (3.3) is used to bound the solution, and (3.4) to bound its gradient. As a simple particular case of these assumptions, we can take $\Lambda_T(a) = K_T(1 + a)$ and $\Gamma_{T,R}(a) = M_{T,R}(1 + a)$ with K_T and $M_{T,R}$ constants (see (3.20), (3.21) and Remark 3.4). With these choices, (3.9) and (3.10) read

$$||u(t, \cdot)||_{L^\infty(\mathbb{R}^N)} \le (1 + ||u_0||_{L^\infty(\mathbb{R}^N)})e^{K_T t} - 1$$
$$||Du(t, \cdot)||_{L^\infty(\mathbb{R}^N)} \le (1 + ||Du_0||_{L^\infty(\mathbb{R}^N)})e^{2NM_{T,R}t} - 1\,,$$

which are quite classical estimates. Note that if this choice of $\Gamma_{T,R}$ in (3.4) is usual (see [12] for $\lambda = 2$, [18] for $\lambda \in]1, 2[$ and [3] for the pure Hamilton-Jacobi equation — *i.e.* without g_λ), Assumption (3.3) is, even with the preceding choice of Λ_T, less usual in the framework of Hamilton-Jacobi equations. To ensure global existence, this hypothesis is in general replaced by a bound on $F(t, x, 0, 0)$ and by the assumption that F is nonincreasing with respect to s (see the preceding references). Assumption (3.3) with $\Lambda_T(a) = K_T(1 + a)$ however appears in [16] in the case of parabolic equations (*i.e.* $\lambda = 2$).
In their general form, Assumptions (3.3) and (3.4) do not seem common in the literature; however, they are completely natural with respect to the technique we use here. They allow to consider, for example, $F(t, x, u, \nabla u) = u^2 \ln(1 + |\nabla u|^2)$.

We now turn to the proof of Theorem 3.1.

3.1.2 L^∞ estimates and uniqueness

The following proposition gives the key estimate, both for Hamilton-Jacobi equations and for scalar conservation laws; it relies on Theorem 2.2 in an essential way. This estimate still holds true in a more general case (precisely, in the framework of viscosity solution, with less regular solutions; see Subsection 3.2 for a definition): see Proposition 5.1 in Appendix. We choose to present below the estimate in the smooth case because we look here for regular solutions.

Proposition 3.1 *Let $\lambda \in]0, 2[$, $T > 0$ and $G \in C(]0, T[\times\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ be such that, for all $R > 0$, $\nabla_\xi G$ is bounded on $]0, T[\times\mathbb{R}^N \times [-R, R] \times B_R$. We also assume that*

$$\begin{array}{c}\text{there exists } h : [0, \infty[\mapsto]0, \infty[\text{ continuous nondecreasing} \\ \text{such that } \int_0^\infty \frac{1}{h(a)}\, da = +\infty \text{ and, for all } (t, x, s) \in]0, T[\times\mathbb{R}^N \times \mathbb{R}, \\ \mathrm{sgn}(s)G(t, x, s, 0) \le h(|s|).\end{array} \tag{3.11}$$

If $u \in C_b^2(]a, T[\times\mathbb{R}^N)$ for all $a \in]0, T[$ and satisfies

$$\partial_t u(t, x) + g_\lambda[u(t, \cdot)](x) = G(t, x, u(t, x), \nabla u(t, x)) \quad \text{on} \quad]0, T[\times\mathbb{R}^N\,, \tag{3.12}$$

then, defining $\mathcal{H}(a) = \int_0^a \frac{1}{h(b)}\, db$, we have, for all $0 < t' < t < T$,

$$||u(t, \cdot)||_{L^\infty(\mathbb{R}^N)} \le \mathcal{H}^{-1}\left(t - t' + \mathcal{H}(||u(t', \cdot)||_{L^\infty(\mathbb{R}^N)})\right). \tag{3.13}$$

Proof of Proposition 3.1
Let $a \in]0, T[$. Since $\partial_t^2 u$ is bounded on $]a/2, T[\times\mathbb{R}^N$ (say by C_a), we have, for all $t \in]a, T[$, all $0 < \tau < a/2$ and all $x \in \mathbb{R}^N$,

$$\begin{array}{rcl} u(t, x) & \le & u(t - \tau, x) + \tau\partial_t u(t, x) + C_a\tau^2 \\ & \le & \sup_{\mathbb{R}^N} u(t - \tau, \cdot) + \tau G(t, x, u(t, x), \nabla u(t, x)) - \tau g_\lambda[u(t, \cdot)](x) + C_a\tau^2. \end{array} \tag{3.14}$$

Fix $t > a$ and assume that $\sup_{\mathbb{R}^N} u(t,\cdot) > 0$. Let $(x_n)_{n\geq 1} \in \mathbb{R}^N$ be a sequence such that $u(t,x_n) \to \sup_{\mathbb{R}^N} u(t,\cdot)$. We have

$$G(t,x_n,u(t,x_n),\nabla u(t,x_n)) \leq G(t,x_n,u(t,x_n),0) + M_t|\nabla u(t,x_n)|$$

where $M_t = \sup\{|\nabla_\xi G(t,x,s,\xi)|\,,\ (x,s,\xi) \in \mathbb{R}^N \times [-R_t,R_t] \times B_{R_t}\}$ with R_t an upper bound of $u(t,\cdot)$ and $\nabla u(t,\cdot)$. For n large enough, $u(t,x_n) > 0$ and thus, by (3.11),

$$G(t,x_n,u(t,x_n),\nabla u(t,x_n)) \leq h(u(t,x_n)) + M_t|\nabla u(t,x_n)| \leq h\left(\sup_{\mathbb{R}^N} u(t,\cdot)\right) + M_t|\nabla u(t,x_n)|.$$

Injected in (3.14), this gives, for all $t \in]a,T[$ and all $0 < \tau < a/2$,

$$u(t,x_n) \leq \sup_{\mathbb{R}^N} u(t-\tau,\cdot) + \tau h\left(\sup_{\mathbb{R}^N} u(t,\cdot)\right) + \tau M_t|\nabla u(t,x_n)| - \tau g_\lambda[u(t,\cdot)](x_n) + C_a\tau^2.$$

By Theorem 2.2, we have $\liminf_{n\to\infty} g_\lambda[u(t,\cdot)](x_n) \geq 0$ and $\nabla u(t,x_n) \to 0$ as $n \to \infty$. Hence, taking the $\limsup_{n\to\infty}$ of the preceding inequality leads to

$$\sup_{\mathbb{R}^N} u(t,\cdot) \leq \sup_{\mathbb{R}^N} u(t-\tau,\cdot) + \tau h\left(\sup_{\mathbb{R}^N} u(t,\cdot)\right) + C_a\tau^2.$$

This has been obtained under the condition that $\sup_{\mathbb{R}^N} u(t,\cdot) > 0$; defining $\Phi(t) = \max(\sup_{\mathbb{R}^N} u(t,\cdot), 0)$, we deduce, whatever the sign of $\sup_{\mathbb{R}^N} u(t,\cdot)$ is, that $\Phi(t) \leq \Phi(t-\tau) + \tau h(\Phi(t)) + C_a\tau^2$, that is to say, for $t \in]a,T[$ and $0 < \tau < a/2$,

$$\frac{\Phi(t) - \Phi(t-\tau)}{\tau} \leq h(\Phi(t)) + C_a\tau.$$

As $\partial_t u$ is bounded on $]a,T[\times\mathbb{R}^N$, it is easy to see that Φ is Lipschitz continuous on $]a,T[$ and this inequality therefore implies $\Phi' \leq h(\Phi)$ almost everywhere on $]0,T[$. Hence, the derivative of the locally Lipschitz continuous function $t \in]0,T[\mapsto \mathcal{H}(\Phi(t))$ (notice that $\mathcal{H}$ is C^1 on $[0,\infty[$) is bounded from above by 1 and, for all $0 < t' < t < T$, $\mathcal{H}(\Phi(t)) \leq t - t' + \mathcal{H}(\Phi(t'))$. Since $\mathcal{H}$ is a nondecreasing bijection $[0,\infty[\mapsto [0,\infty[$, we deduce

$$\sup_{\mathbb{R}^N} u(t,\cdot) \leq \Phi(t) \leq \mathcal{H}^{-1}\left(t - t' + \mathcal{H}(\Phi(t'))\right) \leq \mathcal{H}^{-1}\left(t - t' + \mathcal{H}(||u(t',\cdot)||_{L^\infty(\mathbb{R}^N)})\right).$$

The same reasoning applied to $-u$ (solution to (3.12) with $(t,x,s,\xi) \mapsto -G(t,x,-s,-\xi)$, which also satisfies (3.11), instead of G) gives an upper bound on $\sup_{\mathbb{R}^N}(-u(t,\cdot)) = -\inf_{\mathbb{R}^N} u(t,\cdot)$ and concludes the proof. ■

We deduce from this proposition the following corollary, which implies the uniqueness stated in Theorem 3.1.

Corollary 3.1 *Let $\lambda \in]0,2[$, $T > 0$ and $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. If F satisfies (3.2), then there exists at most one function defined on $]0,T[\times\mathbb{R}^N$ which satisfies (3.6), (3.7) and (3.8).*

Proof of Corollary 3.1
Assume that u and v are two such functions. The difference $w = u - v$ is in $C^2_b(]a,T[\times\mathbb{R}^N)$ for all $a \in]0,T[$ and satisfies

$$\partial_t w(t,x) + g_\lambda[w(t,\cdot)](x) = G(t,x,w(t,x),\nabla w(t,x)) \quad \text{on} \quad]0,T[\times\mathbb{R}^N$$

with

$$\begin{aligned} G(t,x,s,\xi) \quad = \quad & \left(\int_0^1 \partial_s F(t,x,\tau u(t,x) + (1-\tau)v(t,x), \nabla u(t,x))\,d\tau\right) s \\ & + \left(\int_0^1 \nabla_\xi F(t,x,v(t,x),\tau\nabla u(t,x) + (1-\tau)\nabla v(t,x))\,d\tau\right)\cdot\xi. \end{aligned}$$

By (3.2) and the hypotheses on u and v, G is continuous on $]0,T[\times\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ and $\nabla_\xi G$ is bounded on $]0,T[\times\mathbb{R}^N \times [-R,R] \times B_R$ for all $R > 0$. Moreover, G satisfies (3.11) with $h(a) = C(\kappa + a)$ where κ

is any positive number (added so that $h > 0$ on $\mathbb{R}^+$) and C only depends on u, v and the constants in (3.2). For this h, we have $\mathcal{H}(a) = \frac{1}{C}(\ln(\kappa + a) - \ln(\kappa))$ and $\mathcal{H}^{-1}(a) = \kappa e^{Ca} - \kappa$; hence, by Proposition 3.1 we find, for $0 < t' < t < T$, $||w(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \le e^{C(t-t')}(\kappa + ||w(t',\cdot)||_{L^\infty(\mathbb{R}^N)}) - \kappa$. Since $u(t',\cdot) \to u_0$ and $v(t',\cdot) \to u_0$ uniformly on $\mathbb{R}^N$ as $t' \to 0$, we have $||w(t',\cdot)||_{L^\infty(\mathbb{R}^N)} \to 0$ as $t' \to 0$ and we conclude, letting $t' \to 0$ and $\kappa \to 0$ in the preceding inequality, that $w(t,\cdot) = 0$ for all $t \in]0,T[$. ■

3.1.3 $W^{1,\infty}$ estimates and existence

To prove the existence of a solution to (3.1), we first introduce another definition of solution, in the spirit of [26, chapter 15] or [15].

Definition 3.1 *Let $\lambda \in]1,2[$, $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, $T > 0$ and F satisfy (3.2). A* weak solution *to (3.1) on $[0,T]$ is a function $u \in L^\infty(]0,T[\times\mathbb{R}^N)$ such that $\nabla u \in L^\infty(]0,T[\times\mathbb{R}^N)^N$ and, for a.e. $(t,x) \in]0,T[\times\mathbb{R}^N$,*

$$u(t,x) = K_\lambda(t,\cdot) * u_0(x) + \int_0^t K_\lambda(t-s,\cdot) * F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))(x)\,ds\,, \tag{3.15}$$

where K_λ is the kernel associated with g_λ.

The kernel associated with g_λ is $K_\lambda(t,x) = \mathcal{F}^{-1}(\xi \mapsto e^{-t|\xi|^\lambda})(x)$. It is defined so that the solution to $\partial_t v + g_\lambda[v] = 0$ is given by $v(t,x) = K_\lambda(t,\cdot) * v(0,\cdot)(x)$ and (3.15) is simply Duhamel's formula on (3.1). Let us recall the main properties of K_λ (valid for $\lambda \in]0,2[$) which allow in particular to see that each term in (3.15) is well-defined.

$$\begin{gathered} K_\lambda \in C^\infty(]0,\infty[\times\mathbb{R}^N) \text{ and } (K_\lambda(t,\cdot))_{t\to 0} \text{ is an approximate unit} \\ \text{(in particular, } K_\lambda \text{ is nonnegative and, for all } t>0,\ ||K_\lambda(t,\cdot)||_{L^1(\mathbb{R}^N)} = 1), \\ \forall t > 0\,,\ \forall t' > 0\,,\ K_\lambda(t+t',\cdot) = K_\lambda(t,\cdot) * K_\lambda(t',\cdot)\,, \\ \exists \mathcal{K} > 0\,,\ \forall t > 0\,,\ ||\nabla K_\lambda(t,\cdot)||_{L^1(\mathbb{R}^N)} \le \mathcal{K} t^{-1/\lambda} \end{gathered} \tag{3.16}$$

(notice that the nonnegativity of K_λ can be proved from Theorem 2.2 by using the same technique as in the proof of Proposition 3.1). Using the Banach fixed point theorem, it is quite simple to prove the local existence (and uniqueness) of a weak solution to (3.1); its regularity is obtained by the same means. We give in Appendix ideas for the proof of the following theorem and let the reader check the details (see, for example, [15] and [18]).

Theorem 3.2 *Let $\lambda \in]1,2[$, $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ and F satisfy (3.2).*

i) For all $T > 0$, there exists at most one weak solution to (3.1) on $[0,T]$.

ii) A weak solution to (3.1) on $[0,T]$ satisfies (3.6), (3.7) and (3.8).

iii) Let $M \ge ||u_0||_{W^{1,\infty}(\mathbb{R}^N)}$. There exists $T > 0$, only depending on M and the constants in Hypothesis (3.2), such that (3.1) has a weak solution on $[0,T]$.

We now obtain estimates on the gradient of the weak solution, and conclude the proof of Theorem 3.1.

Proposition 3.2 *Let $\lambda \in]1,2[$ and $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. Assume that F satisfies (3.2) and (3.4). If u is a weak solution to (3.1) on $[0,T]$ and $R \ge ||u||_{L^\infty(]0,T[\times\mathbb{R}^N)}$ then (3.10) holds for all $t \in]0,T[$.*

Proof of Proposition 3.2
The proof is very similar to the proof of Proposition 3.1. If $\varphi \in C_b^3(\mathbb{R}^N)$, then a derivation under the integral sign on (2.1) shows that $\partial_i g_\lambda[\varphi] = g_\lambda[\partial_i \varphi]$. Since u satisfies (3.6) and (3.7) (Theorem 3.2), we deduce that

$$\begin{aligned} \partial_t(\partial_i u)(t,x) + g_\lambda[\partial_i u(t,\cdot)](x) &= \partial_{x_i} F(t,x,u(t,x),\nabla u(t,x)) + \partial_s F(t,x,u(t,x),\nabla u(t,x))\partial_i u(t,x) \\ &\quad + \nabla_\xi F(t,x,u(t,x),\nabla u(t,x)) \cdot \nabla(\partial_i u)(t,x). \end{aligned}$$

Let $a \in]0,T[$; the function $\partial_t^2 \partial_i u$ is bounded on $]a/2,T[\times\mathbb{R}^N$ (say by $C_{a,i}$) and thus, for $t \in]a,T[$, $0 < \tau < a/2$ and $x \in \mathbb{R}^N$,

$$\begin{array}{rcl}\partial_i u(t,x) & \leq & \partial_i u(t-\tau,x) + \tau\partial_t\partial_i u(t,x) + C_{a,i}\tau^2 \\ & \leq & \sup_{\mathbb{R}^N} \partial_i u(t-\tau,\cdot) + \tau\partial_{x_i}F(t,x,u(t,x),\nabla u(t,x)) + \tau\partial_s F(t,x,u(t,x),\nabla u(t,x))\partial_i u(t,x) \\ & & +\tau\nabla_\xi F(t,x,u(t,x),\nabla u(t,x))\cdot\nabla(\partial_i u)(t,x) - \tau g_\lambda[\partial_i u(t,\cdot)](x) + C_{a,i}\tau^2. \end{array} \tag{3.17}$$

Assume that $\sup_{\mathbb{R}^N} \partial_i u(t,\cdot) > 0$ and take a sequence $(x_n)_{n\geq 1} \in \mathbb{R}^N$ such that $\partial_i u(t,x_n) \to \sup_{\mathbb{R}^N} \partial_i u(t,\cdot)$. Since $\partial_i u(t,\cdot) \in C_b^2(\mathbb{R}^N)$, Theorem 2.2 gives $\liminf_{n\to\infty} g_\lambda[\partial_i u(t,\cdot)](x_n) \geq 0$ and $\lim_{n\to\infty} \nabla(\partial_i u)(t,x_n) = 0$. For n large enough, $\partial_i u(t,x_n) > 0$ and we can apply (3.17) to $x = x_n$, use (3.4) and (3.2) to bound the terms involving F and then take the $\limsup_{n\to\infty}$ of the resulting inequality; we find

$$\sup_{\mathbb{R}^N} \partial_i u(t,\cdot) \leq \sup_{\mathbb{R}^N} \partial_i u(t-\tau,\cdot) + 2\tau\Gamma_{T,R}\left(\sup_{\mathbb{R}^N} |\nabla u(t,\cdot)|\right) + C_{a,i}\tau^2$$

(recall that R is an upper bound of $||u||_{L^\infty(]0,T[\times\mathbb{R}^N)}$). As in the proof of Proposition 3.1, to obtain an inequality which holds whatever the sign of $\sup_{\mathbb{R}^N} \partial_i u(t,\cdot)$ is, we define $w_{i,+}(t) = \max(\sup_{\mathbb{R}^N} \partial_i u(t,\cdot), 0)$ and we have, for all $t \in]a,T[$ and all $0 < \tau < a/2$,

$$w_{i,+}(t) \leq w_{i,+}(t-\tau) + 2\tau\Gamma_{T,R}\left(||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)}\right) + C_{a,i}\tau^2.$$

This reasoning applied to $-u$ (the function $-F(\cdot,\cdot,-\cdot,-\cdot)$ satisfies (3.2) and (3.4)) leads to the same inequality for $w_{i,-}(t) = \max(\sup_{\mathbb{R}^N}(-\partial_i u(t,\cdot)), 0) = \max(-\inf_{\mathbb{R}^N} \partial_i u(t,\cdot), 0)$. This inequality is therefore also satisfied by $\max(w_{i,+}(t), w_{i,-}(t)) = ||\partial_i u(t,\cdot)||_{L^\infty(\mathbb{R}^N)}$ and, summing on $i = 1,\ldots,N$, we deduce that, for all $t \in]a,T[$ and all $0 < \tau < a/2$,

$$||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \leq ||Du(t-\tau,\cdot)||_{L^\infty(\mathbb{R}^N)} + 2N\tau\Gamma_{T,R}\left(||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)}\right) + \sum_{i=1}^N C_{a,i}\tau^2.$$

Since $t \mapsto ||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)}$ is locally Lipschitz continuous (because $\partial_t\partial_i u$ is bounded on $]a,T[\times\mathbb{R}^N$ for all $a \in]0,T[$), we infer as in the proof of Proposition 3.1 that, for $0 < t' < t < T$,

$$||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \leq (\mathcal{G}_{T,R})^{-1}\left(t - t' + \mathcal{G}_{T,R}(||Du(t',\cdot)||_{L^\infty(\mathbb{R}^N)})\right). \tag{3.18}$$

The function u_0 being Lipschitz continuous, the definition of the derivative and the dominated convergence theorem show that $\partial_i(K(t',\cdot) * u_0) = K(t',\cdot) * \partial_i u_0$. Thanks to Lemma 5.1 in Appendix and (3.16), we have $\partial_i[K_\lambda(t'-s,\cdot) * F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))] = \partial_i K_\lambda(t'-s,\cdot) * F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))$, which is bounded independently of $x \in \mathbb{R}^N$ by an integrable function of $s \in]0,t'[$; we can therefore derivate (3.15) under the integral sign to find

$$\partial_i u(t',x) = K_\lambda(t',\cdot) * \partial_i u_0(x) + \int_0^{t'} \partial_i K_\lambda(t'-s,\cdot) * F(s,\cdot,u(s,\cdot),\nabla u(s,\cdot))(x)\,ds \tag{3.19}$$

and, still using (3.16), we obtain

$$||\partial_i u(t',\cdot)||_{L^\infty(\mathbb{R}^N)} \leq ||\partial_i u_0||_{L^\infty(\mathbb{R}^N)} + ||F(\cdot,\cdot,u,\nabla u)||_\infty \frac{\mathcal{K}t'^{1-\frac{1}{\lambda}}}{1-\frac{1}{\lambda}}.$$

This shows that $\limsup_{t'\to 0} ||Du(t',\cdot)||_{L^\infty(\mathbb{R}^N)} \leq ||Du_0||_{L^\infty(\mathbb{R}^N)}$ and we conclude the proof by letting $t' \to 0$ in (3.18). ■

The proof of the existence and estimates in Theorem 3.1 is then straightforward. Indeed, take u a weak solution to (3.1) on $[0,T]$ given by Theorem 3.2. By (3.3), F satisfies (3.11) with $h = \Lambda_T$; since u satisfies (3.6) and (3.7), Proposition 3.1 shows that, for $0 < t' < t < T$, $||u(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \leq (\mathcal{L}_T)^{-1}(t - t' + \mathcal{L}_T(||u(t',\cdot)||_{L^\infty(\mathbb{R}^N)}))$; but (3.8) holds for u, and we can therefore let $t' \to 0$ to deduce that (3.9) is valid. We have (3.10) by Proposition 3.2. These estimates (3.9) and (3.10) show that the $W^{1,\infty}$ norm of $u(t,\cdot)$

does not explode in finite time; item iii) in Theorem 3.2 then allows to indefinitely extend u ([1]), which gives a global weak solution to (3.1), and thus a global solution in the sense of Theorem 3.1.

Remark 3.2 *As a by-product of this proof of existence, we see that the solution to (3.1) given by Theorem 3.1 also satisfies (3.15), which was not obvious from (3.6)—(3.8).*

Remark 3.3 *The preceding technique also works if g_λ is replaced by a more general operator, provided that it satisfies Theorem 2.2 (in fact, this theorem is only needed for $\varphi \in C_b^\infty(\mathbb{R}^N)$) and that its kernel satisfies (3.16) (for small t and some $\lambda > 1$ in the estimate of the gradient) and (5.1) in Appendix. As interesting and simple examples of such operators, we can mention:*

1) *The laplace operator $-\Delta$ (which corresponds, up to a multiplicative constant, to g_2). Or course, the preceding results are known for semi-linear parabolic equations (at least for classical choices of Λ_T and $\Gamma_{T,R}$).*

2) *Multifractal operators such as in [5], that is to say $\sum_{j=1}^l \alpha_j g_{\lambda_j}$ with $\alpha_j > 0$, $\lambda_j \in]0,2]$ and $\lambda_1 \in]1,2]$. The kernel of this operator is $K_{\lambda_1}(\alpha_1 t, \cdot) * \cdots * K_{\lambda_l}(\alpha_l t, \cdot)$, and it satisfies (3.16) with $\lambda = \lambda_1$.*

3) *Anisotropic operators of the kind $A[\varphi] = \mathcal{F}^{-1}(\sum_{j=1}^N |\xi_j|^{\gamma_j} \mathcal{F}(\varphi)(\xi))$ with $\gamma_j \in]1,2]$ (it comes to take a "γ_j-th derivative" in the j-th direction). This operator is the sum of 1-dimensional operators g_{γ_j} acting on each variable, and thus a formula of the kind of (2.1) can be established, which proves that Theorem 2.2 holds. The kernel of A is $\prod_{j=1}^N k_{\gamma_j}(t, x_j)$, where k_{γ_j} is the kernel of g_{γ_j} in dimension $N = 1$, and it thus satisfies (3.16) with $\lambda = \inf_j(\gamma_j)$.*

We refer the reader to [17] for the kernel properties of other pseudo-differential operators.

3.2 Existence and uniqueness results for $\lambda \in]0,2[$

We show here that, under weaker regularity (but stronger behaviour) assumptions on F and for $\lambda \in]0,2[$, we can still solve (3.1), albeit in the viscosity sense. We assume, in the following, that the Hamiltonian F is continuous with respect to (t, x, s, ξ), locally Lipschitz continuous with respect to (x, s, ξ) and satisfies (3.3). We replace (3.4) by

$$\begin{array}{c}\forall T > 0\,,\ \forall R > 0\,,\ \exists\, \Theta_{T,R} > 0 \text{ such that, for all } (t,x,s,\xi) \in [0,T] \times \mathbb{R}^N \times [-R,R] \times \mathbb{R}^N, \\ \partial_s F(t,x,s,\xi) \le \Theta_{T,R}\end{array} \tag{3.20}$$

$$\begin{array}{c}\forall T > 0\,,\ \forall R > 0\,,\ \exists\, \Xi_{T,R} > 0 \text{ such that, for all } (t,x,s,\xi) \in [0,T] \times \mathbb{R}^N \times [-R,R] \times \mathbb{R}^N, \\ |\nabla_x F(t,x,s,\xi)| \le \Xi_{T,R}(1 + |\xi|).\end{array} \tag{3.21}$$

and (3.2) is relaxed to

$$\begin{array}{c}\forall T > 0, \forall R > 0, \exists C_{T,R} > 0 \text{ such that, for all } (t,x,s,\xi) \in [0,T] \times \mathbb{R}^N \times [-R,R] \times B_R, \\ |F(t,x,s,\xi)| \le C_{T,R}\,,\ |\partial_s F(t,x,s,\xi)| \le C_{T,R}\,,\ |\nabla_\xi F(t,x,s,\xi)| \le C_{T,R}.\end{array} \tag{3.22}$$

In all the preceding inequalities, the derivatives of F are to be understood as the a.e. derivatives of a Lipschitz continuous function; these hypotheses therefore state bounds on F and its Lipschitz constants.

Remark 3.4 *Assumptions (3.20) and (3.21) are stronger than (3.4); they imply this last assumption with, for example, $\Gamma_{T,R}(a) = (\Theta_{T,R} + \Xi_{T,R})(1 + a)$.*

Let us first briefly recall the definition of a viscosity solution to (3.1) (an immediate generalization of the definition given in [18] in the case $\lambda > 1$).

[1] It is the semi-group property of K_λ in (3.16) which tells that, if we glue to $u_{|[0,t_0]}$ a weak solution with initial time t_0 and initial data $u(t_0, \cdot)$, then we construct another weak solution.

Definition 3.2 *Let* $\lambda \in]0,2[$, $u_0 \in C_b(\mathbb{R}^N)$ *and* $F : [0,T[\times\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ *be continuous. A function* $u : [0,T[\times\mathbb{R}^N \mapsto \mathbb{R}$ *is a* viscosity subsolution *to (3.1) if it is bounded upper semi-continuous, if* $u(0,\cdot) \le u_0$ *and if, for all* $(t,x) \in]0,T[\times\mathbb{R}^N$ *and all* $(\alpha,p) \in \mathbb{R}\times\mathbb{R}^N$ *such that there exists* $\sigma > 0$ *and* $r_0 > 0$ *satisfying*

$$u(s,y) \le u(t,x) + \alpha(s-t) + p\cdot(y-x) + \sigma|y-x|^2 + o(s-t) \quad \text{for } y \in B_{r_0}(x) \text{ and } s \in [0,T[, \tag{3.23}$$

we have, for all $r > 0$,

$$\alpha - c_N(\lambda)\int_{B_r} \frac{u(t,x+z)-u(t,x)-p\cdot z}{|z|^{N+\lambda}}\,dz - c_N(\lambda)\int_{\mathbb{R}^N\setminus B_r} \frac{u(t,x+z)-u(t,x)}{|z|^{N+\lambda}}\,dz \le F(t,x,u(t,x),p). \tag{3.24}$$

We similarly define the notion of supersolution *for bounded lower semi-continuous functions by reversing the inequalities (and the sign of* σ*). A function is a* viscosity solution *of (3.1) if it is both a sub- and a supersolution of (3.1).*

Remark 3.5

1. *Since* $u(t,x+z)-u(t,x)-p\cdot z \le \sigma|z|^2$ *on a neighbourhood of* 0*, the first integral term of (3.24) is defined in* $[-\infty,+\infty[$ *(the inequality in fact forbids the case where this term is* $-\infty$*); the second integral term is defined in* $\mathbb{R}$*, because* u *is bounded. Moreover, since* $\int_{B_a\setminus B_b} \frac{p\cdot z}{|z|^{N+\lambda}}\,dz = 0$ *for all* $a > b > 0$*, the quantities in (3.24) in fact do not depend on* $r > 0$*; in particular, if* $\lambda < 1$ *or* $\lambda > 1$*, we can take* $r = 0$ *or* $r = +\infty$*.*

2. *In [18], a couple* $(\alpha,p) \in \mathbb{R}\times\mathbb{R}^N$ *satisfying (3.23) is called a supergradient. The set of all such couples is denoted* $\partial^P u(t,x)$ *and is referred to as the superdifferential of* u *at* (t,x)*. It is the projection on* $\mathbb{R}\times\mathbb{R}^N$ *of the upper jet of* u *at* (t,x) *(see [12] for a definition of semi-jets).*

We can now state our existence and uniqueness result for Lipschitz continuous Hamiltonians and $\lambda \in]0,2[$.

Theorem 3.3 *Let* $\lambda \in]0,2[$ *and* F *be continuous and such that (3.3), (3.20), (3.21) and (3.22) hold true. If* $u_0 \in W^{1,\infty}(\mathbb{R}^N)$*, then there exists a unique viscosity solution of (3.1). Moreover, this solution is Lipschitz continuous with respect to* x *and satisfies, for* $0 < t < T < \infty$*, (3.9) and*

$$||Du(t,\cdot)||_{L^\infty(\mathbb{R}^N)} \le (1 + ||Du_0||_{L^\infty(\mathbb{R}^N)})e^{2N(\Theta_{T,R}+\Xi_{T,R})t} - 1 \tag{3.25}$$

for any $R \ge ||u||_{L^\infty(]0,T[\times\mathbb{R}^N)}$*.*

Remark 3.6

1. *This result can be extended to initial conditions that are merely bounded and uniformly continuous. It suffices to adapt the classical method used for instance in [12]. Notice that, in this case, the Lipschitz continuity of the solution is no longer true.*

2. *As in Remark 3.3, this theorem also holds for more general operators* g_λ*.*

3. *Estimate (3.25) is exactly (3.10) when we take, as in Remark 3.4,* $\Gamma_{T,R}(a) = (\Theta_{T,R} + \Xi_{T,R})(1+a)$*.*

4. *The conclusions of this theorem are the same as the ones of Theorem 3 and Lemma 2 in [18], but the assumptions are more general; see the discussion following Theorem 3.1. Moreover, the* $W^{1,\infty}$ *estimate (3.25) involves a norm that is slightly different from the one used in [18].*

5. *Perron's method. As mentioned in the introduction, Theorem 3.3 can be proved by using Perron's method. In this case, "natural" barriers are to be considered for large time, namely* $t \mapsto \pm(\mathcal{L}_T)^{-1}(t + \mathcal{L}_T(\|u_0\|_{L^\infty(\mathbb{R}^N)}))$*. For small time, in order to ensure that the initial condition is fulfilled, classical barriers of the form* $t \mapsto u_0(x) \pm Ct$ *can be used for* C *large enough.*

Proof of Theorem 3.3
We prove the existence result by regularizing F and using a vanishing viscosity method based on the solution given by Theorem 3.1.

Step 1: regularization of F.
Let $h_\varepsilon : \mathbb{R}^d \to \mathbb{R}^d$ be defined by $h_\varepsilon(z) = \max(1 - \frac{\varepsilon}{|z|}, 0)z$. h_ε is at distance ε of the identity function, null on B_ε and $|h_\varepsilon| \leq |\cdot| - \varepsilon$ on $\mathbb{R}^d \backslash B_\varepsilon$; in particular,

$$\text{for all } z \in \mathbb{R}^d \text{ and } |z'| \leq \varepsilon, \quad |h_\varepsilon(z - z')| \leq |z|. \tag{3.26}$$

We define $F_\varepsilon : \mathbb{R}^{2N+2} \to \mathbb{R}$ by $F_\varepsilon(t, x, s, \xi) = F(h_\varepsilon(t)^+, x, h_{2\varepsilon}(s), h_\varepsilon(\xi))$; let $(t, x, s, \xi) \mapsto \rho_\varepsilon(t, x, s, \xi)$ be a classical regularizing kernel such that $\text{supp}(\rho_\varepsilon) \subset B_\varepsilon$ and define $\tilde{F}_\varepsilon = F_\varepsilon * \rho_\varepsilon$. In dimension $d = 1$, $h'_{2\varepsilon}$ takes its values in $[0, 1]$, and thus $\tilde{F}_\varepsilon$ satisfies (3.20) and (3.21) with the same constants as F (thanks to (3.26)). It therefore satisfies (3.4) with $\Gamma_{T,R}(a) = (\Theta_{T,R} + \Xi_{T,R})(1 + a)$ (see Remark 3.4). We also have (3.2) for $\tilde{F}_\varepsilon$ with $\tilde{C}_{T,R,k} = \sup_{|\alpha| \leq k} ||\partial^\alpha \rho_\varepsilon||_{L^1} \sup_{[0,T] \times \mathbb{R}^N \times [-R,R] \times B_R} |F|$ (this quantity, which depends on ε, is finite thanks to (3.22)).
Property (3.3) for $\tilde{F}_\varepsilon$ is slightly less obvious. Let $(t, x, s) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}$ and $|t'| \leq \varepsilon$, $|x'| \leq \varepsilon$, $|s'| \leq \varepsilon$, $|\xi'| \leq \varepsilon$. In the case $|s| \geq \varepsilon$, we have $\text{sgn}(s) = \text{sgn}(s - s') = \text{sgn}(h_{2\varepsilon}(s - s'))$ since $|s'| \leq \varepsilon$, and thus, by (3.3) and (3.26),

$$\begin{aligned} \text{sgn}(s)F_\varepsilon(t - t', x - x', s - s', 0 - \xi') &= \text{sgn}(h_{2\varepsilon}(s - s'))F(h_\varepsilon(t - t')^+, x - x', h_{2\varepsilon}(s - s'), 0) \\ &\leq \Lambda_T(|h_{2\varepsilon}(s - s')|) \\ &\leq \Lambda_T(|s|). \end{aligned}$$

In the case $|s| \leq \varepsilon$, (3.22) gives

$$\begin{aligned} |F_\varepsilon(t - t', x - x', s - s', 0 - \xi') &- F(h_\varepsilon(t - t')^+, x - x', s, 0)| \\ &= |F(h_\varepsilon(t - t')^+, x - x', 0, 0) - F(h_\varepsilon(t - t')^+, x - x', s, 0)| \\ &\leq \varepsilon C_{T,1} \end{aligned}$$

and therefore

$$\begin{aligned} \text{sgn}(s)F_\varepsilon(t - t', x - x', s - s', 0 - \xi') &\leq \text{sgn}(s)F(h_\varepsilon(t - t')^+, x - x', s, 0) + \varepsilon C_{T,1} \\ &\leq \Lambda_T(|s|) + \varepsilon C_{T,1}. \end{aligned}$$

In any cases, we have $\text{sgn}(s)F_\varepsilon(t - t', x - x', s - s', 0 - \xi') \leq \Lambda_T(|s|) + \varepsilon C_{T,1}$. Multiplying this inequality by $\rho_\varepsilon(t', x', s', \xi')$ and integrating on (t', x', s', ξ') shows that (3.3) holds for $\tilde{F}_\varepsilon$ with $\Lambda_T(a) + \varepsilon C_{T,1}$ instead of $\Lambda_T(a)$.

To sum up this step, we have found a regularization $\tilde{F}_\varepsilon$ of F which converges locally uniformly to F and satisfies (3.2), (3.4) with $\Gamma_{T,R}(a) = (\Theta_{T,R} + \Xi_{T,R})(1 + a)$ (independent of ε) and (3.3) with a function $\Lambda_T^\varepsilon = \Lambda_T + \varepsilon C_{T,1}$ which uniformly converges, as $\varepsilon \to 0$, to Λ_T.

Step 2: passing to the limit.
We take $\lambda \in]0, 2[$ and $\mu \in]1, 2[$. Applying Theorem 3.1 and Remark 3.3, we find a smooth solution u^ε of

$$\left\{ \begin{array}{ll} \partial_t u^\varepsilon(t, x) + g_\lambda[u^\varepsilon(t, \cdot)](x) + \varepsilon g_\mu[u^\varepsilon(t, \cdot)](x) = \tilde{F}_\varepsilon(t, x, u^\varepsilon(t, x), \nabla u^\varepsilon(t, x)) & t > 0\,,\ x \in \mathbb{R}^N\,, \\ u^\varepsilon(0, x) = u_0(x) & x \in \mathbb{R}^N \end{array} \right. \tag{3.27}$$

in the sense of (3.6), (3.7) and (3.8) (notice that, if $\lambda > 1$, there is no need to introduce the term εg_μ in this equation). Since, for $0 < \varepsilon \leq 1$, $\tilde{F}_\varepsilon$ satisfies (3.3) with $\Lambda_T(a) + C_{T,1}$ instead of $\Lambda_T(a)$, the theorem gives estimates on u^ε and ∇u^ε which do not depend on ε.

These estimates and (3.22) show that $\tilde{F}_\varepsilon(\cdot,\cdot,u^\varepsilon,\nabla u^\varepsilon)$ is bounded on $]0,T[\times\mathbb{R}^N$ independently of ε. The integral representation (3.15) reads here (see Remark 3.3)

$$\begin{aligned} u^\varepsilon(t,x) &= K_\lambda(t,\cdot)*K_\mu(\varepsilon t,\cdot)*u_0(x) \\ &\quad + \int_0^t K_\lambda(t-s,\cdot)*K_\mu(\varepsilon(t-s),\cdot)*\tilde{F}_\varepsilon(s,\cdot,u^\varepsilon(s,\cdot),\nabla u^\varepsilon(s,\cdot))(x)\,ds\,, \end{aligned}$$

and (3.16) thus gives $||u^\varepsilon(t,\cdot)-K_\lambda(t,\cdot)*K_\mu(\varepsilon t,\cdot)*u_0||_{L^\infty(\mathbb{R}^N)} \leq Ct$ with C not depending on ε. Since $(K_\lambda(t,\cdot))_{t\to 0}$ and $(K_\mu(t,\cdot))_{t\to 0}$ are approximate units and $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, we easily see that $K_\lambda(t,\cdot)*K_\mu(\varepsilon t,\cdot)*u_0(x) \to u_0(x)$ as $t\to 0$, uniformly with respect to $x\in\mathbb{R}^N$ and $\varepsilon\in]0,1]$. Hence, $u^\varepsilon(t,x)\to u_0(x)$ as $t\to 0$, uniformly with respect to $(x,\varepsilon)\in\mathbb{R}^N\times]0,1]$, and the relaxed upper limit $\limsup^* u^\varepsilon(t,x) = \limsup_{\varepsilon\to 0,(s,y)\to(t,x)} u^\varepsilon(s,y)$ coincides with u_0 at $t=0$. So does the relaxed lower limit $\liminf_* u^\varepsilon = -\limsup^*(-u^\varepsilon)$.
Remark that u^ε is a viscosity solution of (3.27). Since $\tilde{F}_\varepsilon \to F$ locally uniformly, an easy adaptation of the stability theorem of [18] shows that $\limsup^* u^\varepsilon$ is a viscosity subsolution of (3.1) and that $\liminf_* u^\varepsilon$ is a viscosity supersolution of (3.1). The assumptions ensure that the comparison principle holds true for (3.1) (still a straightforward generalization of [18] to the case $\lambda\in]0,2[$). Thus, $\limsup^* u^\varepsilon(0,x) = u_0(x) = \liminf_* u^\varepsilon(0,x)$ implies $\limsup^* u^\varepsilon \leq \liminf_* u^\varepsilon$ and we conclude that u^ε locally uniformly converges to $u = \limsup^* u^\varepsilon = \liminf_* u^\varepsilon$, a viscosity solution to (3.1); the estimates on u stated in the theorem are obtained by passing to the limit in the estimates on u^ε. To finish with, we recall that the comparison principle ensures that the solution we have just constructed is unique in the class of viscosity solutions which satisfy $u(0,\cdot)=u_0$. ■

Since we have proved in Step 2 that a vanishing regularization gives a solution to (3.1), we can now wonder if it is possible to obtain a rate of convergence. The next theorem answers this question.

Theorem 3.4 *Let $(\lambda,\mu)\in]0,2[$ and F be continuous and such that (3.3), (3.20), (3.21) and (3.22) hold true. Let $u_0\in W^{1,\infty}(\mathbb{R}^N)$, u be the viscosity solution of (3.1) and, for $\varepsilon>0$, u^ε be the viscosity solution of*

$$\begin{cases} \partial_t u^\varepsilon(t,x) + g_\lambda[u^\varepsilon(t,\cdot)](x) + \varepsilon g_\mu[u^\varepsilon(t,\cdot)](x) = F(t,x,u^\varepsilon(t,x),\nabla u^\varepsilon(t,x)) & t>0\,,\ x\in\mathbb{R}^N\,, \\ u^\varepsilon(0,x) = u_0(x) & x\in\mathbb{R}^N. \end{cases} \tag{3.28}$$

Then, for all $T>0$,

$$||u^\varepsilon - u||_{C_b([0,T]\times\mathbb{R}^N)} = \begin{cases} \mathcal{O}(\varepsilon) & \text{if } \mu<1\,, \\ \mathcal{O}(\varepsilon|\ln(\varepsilon)|) & \text{if } \mu=1\,, \\ \mathcal{O}(\varepsilon^{1/\mu}) & \text{if } \mu>1. \end{cases}$$

Remark 3.7 *1. As the preceding results, this theorem is valid for more general operators g_λ, and also for $\mu=2$. Moreover, the conclusion still holds if we remove g_λ from both equations (in this case, (3.1) is a pure Hamilton-Jacobi equation).*

2. These rates of convergence are optimal for any $\mu\in]0,2[$ (take $F=0$, remove g_λ, choose $u_0(x) = \max(1-|x|,0)$ and compare $u^\varepsilon(1,0)-u(1,0) = K_\mu(\varepsilon,\cdot)*u_0(0)-1$ thanks to the formula $K_\mu(\varepsilon,x) = \varepsilon^{-N/\mu}K_\mu(1,\varepsilon^{-1/\mu}x)$ and to the property $K_\mu(1,x)\sim C|x|^{-N-\mu}$ as $|x|\to\infty$).

Proof of Theorem 3.4
The proof relies on the same technique as in [18], with modifications due to the presence of g_λ and to the fact that μ can be equal to or less than 1.
By a classical change of unknown function, (3.20) allows to reduce to the case where F is nonincreasing with respect to s. Let $M = \sup_{[0,T[\times(\mathbb{R}^N)^2}\{u(t,x)-u^\varepsilon(t,y)-|x-y|^2/2\alpha-\beta|x|^2/2-\eta t-\gamma/(T-t)\}$, where α, β, η are positive and $\gamma\in]0,1]$. We want to prove that, for appropriate choices of η and γ, M is attained at $t=0$.

Let $\nu > 0$ and $M_\nu = \sup_{[0,T[^2\times(\mathbb{R}^N)^2}\{u(t,x)-u^\varepsilon(s,y)-|x-y|^2/2\alpha-|t-s|^2/2\nu-\beta|x|^2/2-\eta t-\gamma/(T-t)\}$. It is classical that M_ν is attained at some $(t_\nu, s_\nu, x_\nu, y_\nu)$ such that, up to a subsequence, $(t_\nu, s_\nu, x_\nu, y_\nu) \to (\overline{t}, \overline{t}, \overline{x}, \overline{y})$ as $\nu \to 0$, where $(\overline{t}, \overline{x}, \overline{y})$ realizes M. We now assume that $\overline{t} > 0$ and, with good choices of η and γ, we show that this leads to a contradiction.

If $\overline{t} > 0$ then, for ν small enough, $t_\nu > 0$ and $s_\nu > 0$. Let $p_\nu = (x_\nu - y_\nu)/\alpha$; by definition of M_ν, $(\gamma/(T-t_\nu)^2 + (t_\nu - s_\nu)/\nu + \eta, p_\nu + \beta x_\nu)$ is a supergradient of u at (t_ν, x_ν); since u is a subsolution of (3.1), we obtain

$$\begin{aligned}
&\frac{\gamma}{(T-t_\nu)^2} + \frac{t_\nu - s_\nu}{\nu} + \eta \\
&-c_N(\lambda)\int_{B_r} \frac{u(t_\nu, x_\nu+z) - u(t_\nu, x_\nu) - (p_\nu + \beta x_\nu)\cdot z}{|z|^{N+\lambda}}\,dz - c_N(\lambda)\int_{\mathbb{R}^N\setminus B_r} \frac{u(t_\nu, x_\nu+z) - u(t_\nu, x_\nu)}{|z|^{N+\lambda}}\,dz \\
&\qquad \le F(t_\nu, x_\nu, u(t_\nu, x_\nu), p_\nu + \beta x_\nu). \qquad (3.29)
\end{aligned}$$

Similarly, by definition of M_ν we can use $((t_\nu - s_\nu)/\nu, p_\nu)$ in the equation at $(t,x) = (s_\nu, y_\nu)$ which states that u^ε is a supersolution of (3.28) and we obtain

$$\begin{aligned}
&\frac{t_\nu - s_\nu}{\nu} \\
&-c_N(\lambda)\int_{B_r} \frac{u^\varepsilon(s_\nu, y_\nu+z) - u^\varepsilon(s_\nu, y_\nu) - p_\nu\cdot z}{|z|^{N+\lambda}}\,dz - c_N(\lambda)\int_{\mathbb{R}^N\setminus B_r} \frac{u^\varepsilon(s_\nu, y_\nu+z) - u^\varepsilon(s_\nu, y_\nu)}{|z|^{N+\lambda}}\,dz \\
&-\varepsilon c_N(\mu)\int_{B_R} \frac{u^\varepsilon(s_\nu, y_\nu+z) - u^\varepsilon(s_\nu, y_\nu) - p_\nu\cdot z}{|z|^{N+\mu}}\,dz - \varepsilon c_N(\mu)\int_{\mathbb{R}^N\setminus B_R} \frac{u^\varepsilon(s_\nu, y_\nu+z) - u^\varepsilon(s_\nu, y_\nu)}{|z|^{N+\mu}}\,dz \\
&\qquad \ge F(s_\nu, y_\nu, u^\varepsilon(s_\nu, y_\nu), p_\nu). \qquad (3.30)
\end{aligned}$$

We also have, still using the definition of M_ν,

$$u^\varepsilon(s_\nu, y_\nu+z) - u^\varepsilon(s_\nu, y_\nu) + u(t_\nu, x_\nu) - u(t_\nu, x_\nu+z) \ge \frac{\beta|x_\nu|^2}{2} - \frac{\beta|x_\nu+z|^2}{2} = -\beta x_\nu\cdot z - \frac{\beta|z|^2}{2}$$

and, by the estimate on ∇u^ε, $|u^\varepsilon(s_\nu, y_\nu+z) - u^\varepsilon(s_\nu, y_\nu)| \le C|z|$ (here and after, C stands for a positive real number which can change from one line to another but does not depend on ε, r, R, ν, α, β, η or γ). Hence, subtracting (3.29) from (3.30) and using the bounds we have on u and u^ε, we find (for $R \le 1$)

$$\begin{aligned}
&-\frac{\gamma}{(T-t_\nu)^2} - \eta + c_N(\lambda)\frac{\beta}{2}\int_{B_r}\frac{|z|^2}{|z|^{N+\lambda}}\,dz \\
&+C\int_{\mathbb{R}^N\setminus B_r}\frac{1}{|z|^{N+\lambda}}\,dz - \varepsilon c_N(\mu)\int_{B_R}\frac{u^\varepsilon(s_\nu, y_\nu+z) - u^\varepsilon(s_\nu, y_\nu) - p_\nu\cdot z}{|z|^{N+\mu}}\,dz \\
&+C\varepsilon\int_{B_1\setminus B_R}\frac{|z|}{|z|^{N+\mu}}\,dz + C\varepsilon\int_{\mathbb{R}^N\setminus B_1}\frac{1}{|z|^{N+\mu}}\,dz \\
&\qquad \ge F(s_\nu, y_\nu, u^\varepsilon(s_\nu, y_\nu), p_\nu) - F(t_\nu, x_\nu, u(t_\nu, x_\nu), p_\nu + \beta x_\nu). \qquad (3.31)
\end{aligned}$$

Using once again the definition of M_ν, we write

$$u^\varepsilon(s_\nu, y_\nu+z) - u^\varepsilon(s_\nu, y_\nu) - p_\nu\cdot z \ge \frac{|x_\nu - y_\nu|^2 - |x_\nu - y_\nu - z|^2}{2\alpha} - \frac{2(x_\nu - y_\nu)\cdot z}{2\alpha} = -\frac{|z|^2}{2\alpha}.$$

We can therefore bound the integral term containing u^ε in (3.31) and pass to the limit $\nu \to 0$ to obtain

$$\begin{aligned}
&-\frac{\gamma}{(T-\overline{t})^2} - \eta + C\beta\int_{B_r}\frac{|z|^2}{|z|^{N+\lambda}}\,dz + C\int_{\mathbb{R}^N\setminus B_r}\frac{1}{|z|^{N+\lambda}}\,dz \\
&+C\frac{\varepsilon}{\alpha}\int_{B_R}\frac{|z|^2}{|z|^{N+\mu}}\,dz + C\varepsilon\int_{B_1\setminus B_R}\frac{|z|}{|z|^{N+\mu}}\,dz + C\varepsilon \\
&\qquad \ge F(\overline{t}, \overline{y}, u^\varepsilon(\overline{t}, \overline{y}), \overline{p}) - F(\overline{t}, \overline{x}, u(\overline{t}, \overline{x}), \overline{p} + \beta\overline{x})
\end{aligned}$$

where $\overline{p} = (\overline{x}-\overline{y})/\alpha$. Putting $t=0$ and $x=y=0$ in the definition of M, which is attained at $(\overline{t},\overline{x},\overline{y})$, we have $u(\overline{t},\overline{x}) - u^\varepsilon(\overline{t},\overline{y}) - \gamma/(T-\overline{t}) \geq M \geq -\gamma/T$, and thus $u(\overline{t},\overline{x}) \geq u^\varepsilon(\overline{t},\overline{y}) + \gamma/(T-\overline{t}) - \gamma/T \geq u^\varepsilon(\overline{t},\overline{y})$; the function F being nonincreasing with respect to its third variable, we deduce

$$
\begin{aligned}
-\frac{\gamma}{T^2} - \eta + C\beta r^{2-\lambda} + Cr^{-\lambda} + C\frac{\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon\int_{B_1\setminus B_R}\frac{|z|}{|z|^{N+\mu}}\,dz + C\varepsilon \\
\geq F(\overline{t},\overline{y},u(\overline{t},\overline{x}),\overline{p}) - F(\overline{t},\overline{x},u(\overline{t},\overline{x}),\overline{p}+\beta\overline{x}). \qquad (3.32)
\end{aligned}
$$

Using once again the definition of M_ν, we have $\beta|x_\nu|^2 \leq C$ (because $M_\nu \geq M \geq -\gamma/T \geq -1/T$), so that $\beta|\overline{x}| \leq C\sqrt{\beta}$. Moreover, since p_ν satisfies the reverse inequality of (3.23) with u^ε instead of u, and since we have a bound on the spatial Lipschitz constant of u^ε, we find $|p_\nu| \leq C$ and thus $|\overline{p}| \leq C$ and $|\overline{x}-\overline{y}| \leq C\alpha$. Assumptions (3.21) and (3.22) therefore give

$$
-\frac{\gamma}{T^2} - \eta + C\beta r^{2-\lambda} + Cr^{-\lambda} + \frac{C\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon\int_{B_1\setminus B_R}\frac{|z|}{|z|^{N+\mu}}\,dz + C\varepsilon \geq -C\alpha - C\sqrt{\beta}
$$

(we take $\beta \leq 1$). Choosing $\gamma = (C\sqrt{\beta} + C\beta r^{2-\lambda} + Cr^{-\lambda} + \beta)T^2$ (which is in $]0,1]$ if r is large and β is small) and $\eta = C\alpha + \frac{C\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon\int_{B_1\setminus B_R}\frac{|z|}{|z|^{N+\mu}}\,dz + C\varepsilon$ leads to $-\beta \geq 0$, which is the contradiction we sought.

With these choices of γ and η, M is attained at $(0,\overline{x},\overline{y})$ and, for all $(t,x) \in [0,T[\times\mathbb{R}^N$,

$$
u(t,x) - u^\varepsilon(t,x) - \beta\frac{|x|^2}{2} - \eta t - \frac{\gamma}{T-t} \leq u_0(\overline{x}) - u_0(\overline{y}) - \frac{|\overline{x}-\overline{y}|^2}{2\alpha} \leq C\alpha
$$

(we use the fact that u_0 is Lipschitz continuous). Thus,

$$
\begin{aligned}
u(t,x) \quad \leq \quad & u^\varepsilon(t,x) + \beta\frac{|x|^2}{2} + \left(C\alpha + \frac{C\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon\int_{B_1\setminus B_R}\frac{|z|}{|z|^{N+\mu}}\,dz + C\varepsilon\right)T \\
& + \frac{(C\sqrt{\beta} + C\beta r^{2-\lambda} + Cr^{-\lambda} + \beta)T^2}{T-t} + C\alpha.
\end{aligned}
$$

We now let $\beta \to 0$ and then $r \to +\infty$:

$$
u(t,x) \leq u^\varepsilon(t,x) + C\alpha + \frac{C\varepsilon}{\alpha}R^{2-\mu} + C\varepsilon\int_{B_1\setminus B_R}\frac{|z|}{|z|^{N+\mu}}\,dz + C\varepsilon.
$$

If $\mu < 1$ (respectively $\mu = 1$, respectively $\mu > 1$), then $\int_{B_1\setminus B_R}\frac{|z|}{|z|^{N+\mu}}\,dz$ is bounded by C (respectively $C|\ln(R)|$, respectively $CR^{1-\mu}$). A simple optimization with respect to R and then α leads to

$$
u(t,x) \leq u^\varepsilon(t,x) + C\left\{\begin{array}{ll} \varepsilon & \text{if } \mu < 1 \\ \varepsilon|\ln(\varepsilon)| & \text{if } \mu = 1 \\ \varepsilon^{1/\mu} & \text{if } \mu > 1 \end{array}\right.
$$

and we obtain the reverse inequality by exchanging, from the beginning, the roles of u and u^ε. ■

4 Fractal scalar hyperbolic equations

4.1 Existence and uniqueness of a smooth solution

In this section, we come back to the case $\lambda \in]1,2[$ and we handle

$$
\left\{\begin{array}{ll} \partial_t u(t,x) + \mathrm{div}(f(t,x,u(t,x))) + g_\lambda[u(t,\cdot)](x) = h(t,x,u(t,x)) & t>0,\ x\in\mathbb{R}^N, \\ u(0,x) = u_0(x) & x\in\mathbb{R}^N, \end{array}\right. \qquad (4.1)
$$

where $u_0 \in L^\infty(\mathbb{R}^N)$, $f \in C^\infty([0,\infty[\times\mathbb{R}^N\times\mathbb{R})^N$, $h \in C^\infty([0,\infty[\times\mathbb{R}^N\times\mathbb{R})$ and

$$\begin{array}{c}\forall T>0\,,\ \forall R>0\,,\ \forall k\in\mathbb{N}\,, \exists C_{T,R,k} \text{ such that,}\\ \text{for all } (t,x,s)\in[0,T]\times\mathbb{R}^N\times[-R,R] \text{ and all } \alpha\in\mathbb{N}^{N+2} \text{ satisfying } |\alpha|\le k,\\ |\partial^\alpha f(t,x,s)|+|\partial^\alpha h(t,x,s)|\le C_{T,R,k}\,,\end{array} \tag{4.2}$$

$$\begin{array}{c}\forall T>0, \text{ there exists } \Lambda_T:[0,+\infty[\to]0,+\infty[\text{ continuous nondecreasing}\\ \text{such that } \int_0^\infty \frac{1}{\Lambda_T(a)}\,da=+\infty \text{ and, for all } (t,x,s)\in[0,T]\times\mathbb{R}^N\times\mathbb{R}\,,\\ \operatorname{sgn}(s)\Big(h(t,x,s)-\sum_{i=1}^N\partial_{x_i}f_i(t,x,s)\Big)\le\Lambda_T(|s|).\end{array} \tag{4.3}$$

The term $h-\sum_{i=1}^N \partial_{x_i} f_i$ represents a source for (4.1), and an assumption on this source is not unexpected if we want global solutions; this hypothesis with $\Lambda_T(a)=K_T(1+a)$ (and K_T constant), as well as uniform spatial bounds such as in (4.2), also appear in [21] when dealing with the pure scalar conservation law (*i.e.* without g_λ). Here, we prove the following.

Theorem 4.1 *Let $\lambda\in]1,2[$ and $u_0\in L^\infty(\mathbb{R}^N)$; assume that f and h satisfy (4.2) and (4.3). Then there exists a unique solution u to (4.1) in the sense: for all $T>0$,*

$$u\in C_b(]0,T[\times\mathbb{R}^N) \text{ and, for all } a\in]0,T[\,,\ u\in C_b^\infty(]a,T[\times\mathbb{R}^N)\,, \tag{4.4}$$

$$u \text{ satisfies the PDE of (4.1) on }]0,T[\times\mathbb{R}^N\,, \tag{4.5}$$

$$u(t,\cdot)\to u_0 \text{ in } L^\infty(\mathbb{R}^N) \text{ weak-}*, \text{ as } t\to 0. \tag{4.6}$$

We also have Estimate (3.9) on the solution, that is to say: for all $0<t<T<\infty$,

$$||u(t,\cdot)||_{L^\infty(\mathbb{R}^N)}\le(\mathcal{L}_T)^{-1}\left(t+\mathcal{L}_T(||u_0||_{L^\infty(\mathbb{R}^N)})\right) \qquad \text{with} \qquad \mathcal{L}_T(a)=\int_0^a\frac{1}{\Lambda_T(b)}\,db.$$

Remark 4.1 *The proof of uniqueness shows that the solution to (4.1) also satisfies (4.7) below. As a consequence, the convergence in (4.6) also holds in $L^p_{\rm loc}(\mathbb{R}^N)$ for all $p<\infty$.*

As for (3.1), the existence of a solution to (4.1) is obtained via a weak formulation based on Duhamel's formula.

Definition 4.1 *Let $\lambda\in]1,2[$, $u_0\in L^\infty(\mathbb{R}^N)$, $T>0$ and (f,h) satisfy (4.2). A weak solution to (4.1) on $[0,T]$ is a function $u\in L^\infty(]0,T[\times\mathbb{R}^N)$ such that, for a.e. $(t,x)\in]0,T[\times\mathbb{R}^N$,*

$$u(t,x)=K_\lambda(t,\cdot)*u_0(x)-\int_0^t\nabla K_\lambda(t-s,\cdot)*f(s,\cdot,u(s,\cdot))(x)\,ds+\int_0^t K_\lambda(t-s,\cdot)*h(s,\cdot,u(s,\cdot))(x)\,ds. \tag{4.7}$$

Thanks to (3.16), each term in (4.7) is well-defined. As before, a fixed point technique (see [15]) allows to prove the theorem stated below (Theorem 4.2); we leave the details to the interested reader (notice that once it has been proved that weak solutions to (4.1) have one continuous spatial derivative — which is a consequence of a result similar to Proposition 5.2 —, the full regularity of these weak solutions can be seen as a consequence of Theorem 3.2).

Theorem 4.2 *Let $\lambda\in]1,2[$, $u_0\in L^\infty(\mathbb{R}^N)$ and (f,h) satisfy (4.2).*

i) For all $T>0$, there exists at most one weak solution to (4.1) on $[0,T]$.

ii) A weak solution to (4.1) on $[0,T]$ satisfies (4.4), (4.5) and (4.6).

iii) Let $M\ge||u_0||_{L^\infty(\mathbb{R}^N)}$. There exists $T>0$, only depending on M and the constants in (4.2), such that (4.1) has a weak solution on $[0,T]$.

We can now prove the existence and uniqueness result for (4.1).

Proof of Theorem 4.1
Let u be a weak solution to (4.1) on $[0,T]$ in the sense of Definition 4.1. By Theorem 4.2, such a solution exists and satisfies (4.4), (4.5); hence, it satisfies (3.12) with

$$G(t,x,s,\xi)=h(t,x,s)-\sum_{i=1}^{N}\partial_{x_i}f_i(t,x,s)-\partial_s f(t,x,s)\cdot\xi.$$

Since (3.11) holds for G with $h=\Lambda_T$ given by Hypothesis (4.3), we deduce from Proposition 3.1 that, for all $0<t'<t<T$,

$$||u(t,\cdot)||_{L^\infty(\mathbb{R}^N)}\le(\mathcal{L}_T)^{-1}\left(t-t'+\mathcal{L}_T(||u(t',\cdot)||_{L^\infty(\mathbb{R}^N)})\right). \tag{4.8}$$

From (4.7) it is easy to see that $\limsup_{t'\to 0}||u(t',\cdot)||_{L^\infty(\mathbb{R}^N)}\le||u_0||_{L^\infty(\mathbb{R}^N)}$ (the last two terms of (4.7) tend to 0 in $L^\infty(\mathbb{R}^N)$ as $t\to 0$, thanks to (3.16)); hence, letting $t'\to 0$ in (4.8) shows that u satisfies (3.9). In particular, the L^∞ norm of $u(t,\cdot)$ does not explode in finite time and, by iii) in Theorem 4.2, we can indefinitely extend u; this proves the existence part of Theorem 4.1.

It remains to prove the uniqueness of the solution. Let u satisfy (4.4), (4.5) and (4.6) for all $T>0$; take $t_0>0$. The function $u(t_0+\cdot,\cdot)$ belongs, for all $T>0$, to $C_b^\infty([0,T[\times\mathbb{R}^N)$; hence, it satifies (3.6) and (3.8) with $u_0=u(t_0,\cdot)$. Moreover, if we define

$$F(t,x,s,\xi)=h(t_0+t,x,u(t_0+t,x))-\mathrm{div}(f(t_0+t,x,u(t_0+t,x)))$$

(in fact, F does not depend on s or ξ), the function $u(t_0+\cdot,\cdot)$ also satisfies (3.7). It is clear that this F satisfies (3.2), (3.3) and (3.4) (with Λ_T and $\Gamma_{T,R}$ constants) and, therefore, $u(t_0+\cdot,\cdot)$ is the unique solution to (3.1) given by Theorem 3.1; in particular, by Remark 3.2 we have, for all $t>0$,

$$\begin{aligned}u(t_0+t,x) &= K_\lambda(t,\cdot)*u(t_0,\cdot)(x)\\ &\quad+\int_0^t K_\lambda(t-s,\cdot)*\left[h(t_0+s,\cdot,u(t_0+s,\cdot))-\mathrm{div}(f(t_0+s,\cdot,u(t_0+s,\cdot)))\right](x)\,ds\\ &= K_\lambda(t,\cdot)*u(t_0,\cdot)(x)-\int_0^t\nabla K_\lambda(t-s,\cdot)*f(t_0+s,\cdot,u(t_0+s,\cdot))(x)\,ds\\ &\quad+\int_0^t K_\lambda(t-s,\cdot)*h(t_0+s,\cdot,u(t_0+s,\cdot))(x)\,ds.\end{aligned} \tag{4.9}$$

For $t>0$ and $x\in\mathbb{R}^N$ fixed, by (4.6) we have $K_\lambda(t,\cdot)*u(t_0,\cdot)(x)\to K_\lambda(t,\cdot)*u_0(x)$ as $t_0\to 0$; using (4.4) and the dominated convergence theorem, we can let $t_0\to 0$ in the last two terms of (4.9) to see that u satisfies (4.7). Hence, u is a weak solution to (4.1) and, by i) in Theorem 4.2, is unique. ■

Remark 4.2 *Equation (4.1) can also be solved with more general operators g_λ, see Remark 3.3.*

4.2 About the vanishing regularization

Let us say a few things on the behaviour, as $\varepsilon\to 0^+$, of the solution to

$$\left\{\begin{array}{ll}\partial_t u^\varepsilon(t,x)+\mathrm{div}(f(t,x,u^\varepsilon(t,x)))+\varepsilon g_\lambda[u^\varepsilon(t,\cdot)](x)=h(t,x,u^\varepsilon(t,x)) & t>0\,,\ x\in\mathbb{R}^N\,,\\ u^\varepsilon(0,x)=u_0(x) & x\in\mathbb{R}^N\,,\end{array}\right. \tag{4.10}$$

where we still take $\lambda\in]1,2[$, $u_0\in L^\infty(\mathbb{R}^N)$ and (f,h) satisfying (4.2) and (4.3). It has been proved in [13] that, if $h=0$ and f does not depend on (t,x), the solution u^ε to (4.10) converges, as $\varepsilon\to 0$, to the entropy solution u of

$$\begin{cases} \partial_t u(t,x) + \operatorname{div}(f(t,x,u(t,x))) = h(t,x,u(t,x)) & t > 0\,,\ x \in \mathbb{R}^N\,, \\ u(0,x) = u_0(x) & x \in \mathbb{R}^N. \end{cases} \tag{4.11}$$

The key argument is the obtention, via a splitting method, of entropy inequalities for (4.10); this method can be generalized to some cases where f and h depend on (t,x) (see [14]) but, in any cases, it is quite technical.

Thanks to formula (2.1), we have a trivial proof of these entropy inequalities, via the following lemma.

Lemma 4.1 *Let $\lambda \in]0,2[$, $\varphi \in C_b^2(\mathbb{R}^N)$ and $\eta \in C^2(\mathbb{R})$ be a convex function. Then $g_\lambda[\eta(\varphi)] \leq \eta'(\varphi) g_\lambda[\varphi]$.*

Proof of Lemma 4.1
Since η is convex, we have $\eta(b) - \eta(a) \geq \eta'(a)(b-a)$. Hence,

$$\eta(\varphi(x+z)) - \eta(\varphi(x)) \geq \eta'(\varphi(x))(\varphi(x+z) - \varphi(x))$$

and

$$\eta(\varphi(x+z)) - \eta(\varphi(x)) - \nabla(\eta(\varphi))(x)\cdot z \geq \eta'(\varphi(x))(\varphi(x+z) - \varphi(x) - \nabla\varphi(x)\cdot z).$$

The conclusion follows from these inequalities and (2.1). ■

Thus, if $\eta \in C^2(\mathbb{R})$ is convex and ϕ is such that $\partial_s \phi(t,x,s) = \eta'(s)\partial_s f(t,x,s)$, multiplying the PDE of (4.10) by $\eta'(u^\varepsilon(t,x))$ gives (recall that all the functions, including u^ε, are regular)

$$\begin{aligned} \partial_t(\eta(u^\varepsilon))(t,x) + \varepsilon g_\lambda[\eta(u^\varepsilon(t,\cdot))](x) \quad &\leq \quad \eta'(u^\varepsilon(t,x)) \left(h(t,x,u^\varepsilon(t,x)) - \sum_{i=1}^N \partial_{x_i} f_i(t,x,u^\varepsilon(t,x)) \right) \\ &\quad -\operatorname{div}(\phi(t,x,u^\varepsilon(t,x))) + \sum_{i=1}^N \partial_{x_i}\phi_i(t,x,u^\varepsilon(t,x))\,, \end{aligned} \tag{4.12}$$

which is exactly the entropy inequality for (4.10). Once this inequality is established, the doubling variable technique of [21] (used in [13]) shows that, for all $T > 0$ and as $\varepsilon \to 0$, $u^\varepsilon \to u$ in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$.

It is also possible, if the initial condition u_0 is in $L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$, to obtain a rate of convergence: $\mathcal{O}(\varepsilon^{1/\lambda})$ in $C([0,T]; L^1(\mathbb{R}^N))$; this is well-known for $\lambda = 2$ (see [22]) and has been done for $\lambda \in]1,2[$, $h = 0$ and $f(t,x,u) = f(u)$ in [13]. However, to obtain such a rate of convergence we must first establish L^1 and BV estimates on u^ε, which demands additional hypotheses on f and h (some integrability properties with respect to x); we refer the reader to [14] for a set of suitable hypotheses. Once these estimates are established, the proof of the rate of convergence is made as in [22] or [13] by using (4.12).

5 Appendix

5.1 A technical Lemma

Lemma 5.1 *If $f \in C^1(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N)$ and $g \in L^\infty(\mathbb{R}^N)$, then $f * g \in C^1(\mathbb{R}^N)$ and $\nabla(f*g) = \nabla f * g$.*

Proof of Lemma 5.1
We have not assumed that $\nabla f(x-y)$ is bounded locally uniformly in x by some integrable function of y; hence, we cannot directly use a theorem of derivation under the integral sign.

Let $n \geq 1$ and define $g_n = g\mathbf{1}_{B_n}$; for all $x \in \mathbb{R}^N$, $f * g_n(x) = \int_{B_n} f(x-y)g(y)\,dy \to f * g(x)$ as $n \to \infty$. Since $f \in C^1(\mathbb{R}^N)$, a derivation under the integral sign shows that $f * g_n \in C^1(\mathbb{R}^N)$ with $\nabla(f * g_n) = \nabla f * g_n$. But, for all $|x| \leq R$,

$$|\nabla f * g_n(x) - \nabla f * g(x)| \leq ||g||_{L^\infty(\mathbb{R}^N)} \int_{\{|y|\geq n\}} |\nabla f(x-y)|\,dy \leq ||g||_{L^\infty(\mathbb{R}^N)} \int_{\{|z|\geq n-R\}} |\nabla f(z)|\,dz$$

(if $|x| \leq R$ and $|y| \geq n$, then $|x-y| \geq |y| - |x| \geq n - R$); hence, since $\nabla f \in L^1(\mathbb{R}^N)$, we have $\nabla(f * g_n) = \nabla f * g_n \to \nabla f * g$ locally uniformly on $\mathbb{R}^N$, which concludes the proof of the lemma. ■

5.2 Generalizations of Theorem 2.2 and Proposition 3.1

In this subsection, we state and prove generalizations of Theorem 2.2 and Proposition 3.1. Roughly speaking, we show that u needs not be in C_b^2 but only in C_b; in this case, the operator g_λ and Equation (3.12) must be understood in the viscosity sense.
For usc functions $\phi :]0,T[\to \mathbb{R}$, the notion of viscosity supergradient is used in order to define viscosity subsolutions of $\phi' = h(\phi)$. The notion of upper semi-continuous envelope of locally bounded functions is also used in the following. The definitions of viscosity supergradient, viscosity solution of $\phi' = h(\phi)$ and upper semi-continuous envelope can be found in [12].

Theorem 5.1 *Let $\lambda \in]0,2[$ and $v \in C_b(]0,T[\times\mathbb{R}^N)$. Let ϕ denote the upper semi-continuous envelope of the function $\sup_{x\in\mathbb{R}^N} v(\cdot,x)$. Then for any viscosity supergradient α of ϕ at $t \in]0,T[$, there exist $t_n \to t$, $\alpha_n \to \alpha$, and $x_n, p_n \in \mathbb{R}^N$ such that*

$$v(t_n,x_n) \to \phi(t) \quad \textit{and} \quad (\alpha_n,p_n) \in \partial^P v(t_n,x_n) \quad \textit{and} \quad p_n \to 0 \quad \textit{and} \quad \liminf_{n\to\infty} g_\lambda[v(t_n,\cdot)](x_n) \geq 0.$$

Proof of Theorem 5.1
By definition of viscosity supergradient, there exists $\psi \in C^1(]0,T[)$ such that $\phi - \psi$ attains a global maximum at t and $\alpha = \psi'(t)$. Then for any $(s,x) \in]0,T[\times\mathbb{R}^N$, we have:

$$v(s,x) - \psi(s) \leq \phi(s) - \psi(s) \leq \phi(t) - \psi(t).$$

Next, for any $\varepsilon > 0$, consider $(t_\varepsilon, x_\varepsilon) \in]0,T[\times\mathbb{R}^N$ such that $\phi(t) < v(t_\varepsilon,x_\varepsilon) + \varepsilon/2$ and $t_\varepsilon \to t$ as $\varepsilon \to 0$. We can also ensure that $\psi(t) \geq \psi(t_\varepsilon) - \varepsilon/2$. Combining these facts yields:

$$\sup_{(s,x)\in]0,T[\times\mathbb{R}^N} (v(s,x) - \psi(s)) < v(t_\varepsilon,x_\varepsilon) - \psi(t_\varepsilon) + \varepsilon.$$

We then apply Borwein and Preiss' minimization principle (see for instance [10]) and get $(s_\varepsilon, y_\varepsilon)$ and $(r_\varepsilon, z_\varepsilon)$ such that

$$|(r_\varepsilon,z_\varepsilon) - (t_\varepsilon,x_\varepsilon)| < \varepsilon^{1/4} \quad \text{and} \quad |(s_\varepsilon,y_\varepsilon) - (r_\varepsilon,z_\varepsilon)| < \varepsilon^{1/4}$$
$$\text{and} \quad \sup_{(s,x)\in]0,T[\times\mathbb{R}^N} (v(s,x) - \psi(s)) \leq v(s_\varepsilon,y_\varepsilon) - \psi(s_\varepsilon) + \varepsilon$$

and such that $(s_\varepsilon, y_\varepsilon)$ is the unique point realizing the maximum of the perturbed function $(t,x) \mapsto v(t,x) - \psi(t) - \sqrt{\varepsilon}(t-r_\varepsilon)^2 - \sqrt{\varepsilon}|x-z_\varepsilon|^2$. This implies that $(\psi'(s_\varepsilon) + 2\sqrt{\varepsilon}(s_\varepsilon - r_\varepsilon), 2\sqrt{\varepsilon}(y_\varepsilon - z_\varepsilon)) \in \partial^P v(s_\varepsilon, y_\varepsilon)$. Define $\alpha_\varepsilon = \psi'(s_\varepsilon) + 2\sqrt{\varepsilon}(s_\varepsilon - r_\varepsilon)$ and $p_\varepsilon = 2\sqrt{\varepsilon}(y_\varepsilon - z_\varepsilon)$. They verify $\alpha_\varepsilon \to \alpha$ and $p_\varepsilon \to 0$ as $\varepsilon \to 0$. Moreover, $v(s_\varepsilon, y_\varepsilon) \to \phi(t)$ and $s_\varepsilon \to t$. It only remains to prove that $\liminf_{\varepsilon\to 0} g_\lambda[v(s_\varepsilon,\cdot)](y_\varepsilon) \geq 0$ by using Fatou's Lemma. First, notice that

$$v(s_\varepsilon, y_\varepsilon + z) - v(s_\varepsilon, y_\varepsilon) \leq \phi(s_\varepsilon) - v(s_\varepsilon, y_\varepsilon)$$

and since ϕ is upper semi-continuous and $v(s_\varepsilon, y_\varepsilon) \to \phi(t)$, the upper limit of the right-hand side is nonpositive. Secondly,

$$\frac{v(s_\varepsilon, y_\varepsilon + z) - v(s_\varepsilon, y_\varepsilon)}{|z|^{N+\lambda}} \le \frac{2\|v\|_\infty}{|z|^{N+\lambda}} \in L^1(\mathbb{R}^N \setminus B_1),$$
$$\frac{v(s_\varepsilon, y_\varepsilon + z) - v(s_\varepsilon, y_\varepsilon) - p_\varepsilon \cdot z}{|z|^{N+\lambda}} \le \frac{\sqrt{\varepsilon}}{|z|^{N+\lambda-2}} \in L^1(B_1).$$

Now choose $\varepsilon = 1/n$ and $(t_n, \alpha_n, x_n, p_n) = (s_{\varepsilon_n}, \alpha_{\varepsilon_n}, y_{\varepsilon_n}, p_{\varepsilon_n})$ satisfies the desired properties. ■

Proposition 5.1 *Let $\lambda \in]0, 2[$, $T > 0$ and $G \in C(]0, T[\times\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ be such that (3.11) is satisfied and G is locally Lipschitz continuous w.r.t. ξ, locally in (t, s) and uniformly in x. Then any viscosity solution of (3.12) satisfies for any $0 < t' < t < T$:*

$$||u(t,\cdot)||^*_{L^\infty(\mathbb{R}^N)} \le \mathcal{H}^{-1}\left(t - t' + \mathcal{H}\left(||u(t',\cdot)||^*_{L^\infty(\mathbb{R}^N)}\right)\right)$$

where $||u(s,\cdot)||^*_{L^\infty(\mathbb{R}^N)} = \limsup_{\tau\to s} ||u(\tau,\cdot)||_{L^\infty(\mathbb{R}^N)}$.

Proof of Proposition 5.1
Let us denote $\phi(t) = ||u(t,\cdot)||^*_{L^\infty(\mathbb{R}^N)}$. Suppose we have proved that ϕ is a viscosity subsolution of $w' = h(w)$ on $]0, T[$. Then the function $\mathcal{H}(\phi(t)) - t$ is a viscosity subsolution of $w' = 0$ (recall that $\mathcal{H}$ is C^1 and nondecreasing). This implies that $\mathcal{H}(\phi(t)) - t$ is nonincreasing and, since $\mathcal{H}$ is a nondecreasing bijection $[0, +\infty[\to [0, +\infty[$, we get the desired *a priori* estimate on u.
It remains to prove that ϕ is a viscosity subsolution of $w' = h(w)$ on $]0, T[$. It is a consequence of Theorem 5.1 applied to $v = |u|$. Let α be a viscosity supergradient of ϕ and consider $t_n \to t$, $\alpha_n \to \alpha$, and $x_n, p_n \in \mathbb{R}^N$ given by Theorem 5.1. We have to prove that $\alpha \le h(\phi(t))$. We distinguish two cases. Suppose first that there exists a sequence $n_k \to \infty$ such that $v(t_{n_k}, x_{n_k}) = u(t_{n_k}, x_{n_k})$. Then $(\alpha_{n_k}, p_{n_k}) \in \partial^P u(t_{n_k}, x_{n_k})$ and, since u is a viscosity subsolution of (3.12), we get:

$$\alpha_{n_k} + g_\lambda[u(t_{n_k},\cdot)](x_{n_k}) \le G(t_{n_k}, x_{n_k}, u(t_{n_k}, x_{n_k}), p_{n_k}).$$

As $k \to \infty$, we have $u(t_{n_k}, x_{n_k}) \to \phi(t)$ and $p_{n_k} \to 0$. We can use the local Lipschitz continuity of G with respect to ξ and find:

$$\alpha_{n_k} + g_\lambda[u(t_{n_k},\cdot)](x_{n_k}) \le h(u(t_{n_k}, x_{n_k})) + M|p_{n_k}|$$

for M independent of k. As k goes to $+\infty$, we conclude in the first case that $\alpha \le h(\phi(t))$ by using the fact that $u(t_{n_k}, x_{n_k}) \to \phi(t)$ and that $g_\lambda[u(t_{n_k},\cdot)](x_{n_k}) \ge g_\lambda[v(t_{n_k},\cdot)](x_{n_k})$ (because $v(t_{n_k}, x_{n_k}) = u(t_{n_k}, x_{n_k})$ and $v(t_{n_k}, x_{n_k} + z) \ge u(t_{n_k}, x_{n_k} + z)$), so that $\liminf_{k\to\infty} g_\lambda[u(t_{n_k},\cdot)](x_{n_k}) \ge 0$. In the second case, for n large enough, $v(t_n, x_n) = -u(t_n, x_n)$. Then $(-\alpha_n, -p_n) \in \partial_P u(t_n, x_n)$ and we can argue similarly, by using the fact that u is a viscosity supersolution of (3.12), to conclude that we also have $\alpha \le h(\phi(t))$. ■

5.3 Ideas for the proof of Theorem 3.2

We need the following additional property on K_λ:

$$t \in]0, \infty[\mapsto K_\lambda(t,\cdot) \in L^1(\mathbb{R}^N) \text{ is continuous.} \tag{5.1}$$

This continuity is a consequence of the regularity of K_λ and of the homogeneity property $K_\lambda(t, x) = t^{-N/\lambda}K_\lambda(1, t^{-1/\lambda}x)$ which shows that, if A is a compact subset of $]0, \infty[$, then $(K_\lambda(t,\cdot))_{t\in A}$ is equi-integrable at infinity (that is to say, for all $\varepsilon > 0$, there exists $R > 0$ such that, for all $t \in A$, $\int_{\mathbb{R}^N\setminus B_R} |K_\lambda(t, x)|\, dx \le \varepsilon$).

The most difficult task in the proof of Theorem 3.2 is the regularity of the weak solutions. The key result to prove this regularity is the following proposition.

Proposition 5.2 *Let $\lambda \in]1,2[$, $S>0$ and $G:(t,x,\zeta)\in]0,S[\times\mathbb{R}^N\times\mathbb{R}^N \to G(t,x,\zeta)\in\mathbb{R}$ be continuous; we suppose that $\partial_x G$, $\partial_\zeta G$, $\partial_\zeta\partial_x G$ and $\partial_\zeta\partial_\zeta G$ exist and are continuous on $]0,S[\times\mathbb{R}^N\times\mathbb{R}^N$; we also suppose that there exists $\omega:]0,\infty[\to\mathbb{R}^+$ such that, for all $L>0$, G and these derivatives are bounded on $]0,S[\times\mathbb{R}^N\times B_L$ by $\omega(L)$.*
Let $R_0>0$ and $R=(2+\mathcal{K})R_0$ where $\mathcal{K}$ is given by (3.16). Then there exists $T_0>0$ only depending on (λ,R_0,ω) such that, if $T=\inf(S,T_0)$ and $V_0\in L^\infty(\mathbb{R}^N)^N$ satisfies $||V_0||_{L^\infty(\mathbb{R}^N)^N}\le R_0$, there exists a unique $V\in C_b(]0,T[\times\mathbb{R}^N)^N$ bounded by R and such that

$$V(t,x)=K_\lambda(t,\cdot)*V_0(x)+\int_0^t \nabla K_\lambda(t-s,\cdot)*G(s,\cdot,V(s,\cdot))(x)\,ds. \tag{5.2}$$

Moreover, $\partial_x V\in C(]0,T[\times\mathbb{R}^N)^{N^2}$ and, for all $a\in]0,T[$, $||\partial_x V||_{C_b(]a,T[\times\mathbb{R}^N)^{N^2}}\le Ra^{-1/\lambda}$.

Sketch of the proof of Proposition 5.2
We define $E_T=\{V\in C_b(]0,T[\times\mathbb{R}^N)^N \mid t^{1/\lambda}\partial_x V\in C_b(]0,T[\times\mathbb{R}^N)^{N^2}\}$ and, for $V\in E_T$, $\Phi_T(V)$ as the right-hand side of (5.2).
Thanks to (5.1), the first term $K_\lambda(t,\cdot)*V_0(x)$ of $\Phi_T(V)$ is continuous in t uniformly with respect to x; since, for t fixed, it is also continuous in x (it is the convolution product of an integrable function and a bounded function), it is continuous in (t,x). The second term of $\Phi_T(V)$ is the convolution product in $\mathbb{R}\times\mathbb{R}^N$ of the integrable function $\nabla K_\lambda(t,x)\mathbf{1}_{]0,T[}(t)$ and the bounded function $G(t,x,V(t,x))\mathbf{1}_{]0,T[}(t)$: it is therefore continuous in (t,x). By Lemma 5.1 and (3.16), we have $\partial_x(K_\lambda(t,\cdot)*V_0)(x)=\partial_x K_\lambda(t,\cdot)*V_0(x)$; since $V\in E_T$, we can differentiate the second term of $\Phi_T(V)$ under the integral sign to obtain

$$\begin{aligned}\partial_x\Phi_T(V)(t,x) &= \partial_x K_\lambda(t,\cdot)*V_0(x)\\ &\quad+\int_0^t \nabla K_\lambda(t-s,\cdot)*\Big[\partial_x G(s,\cdot,V(s,\cdot))+\partial_\zeta G(s,\cdot,V(s,\cdot))\partial_x V(s,\cdot)\Big](x)\,ds.\end{aligned}$$

For $t_0>0$ and $t>0$, we have $\partial_x(K_\lambda(t_0+t,\cdot)*V_0)(x)=K_\lambda(t,\cdot)*(\partial_x K_\lambda(t_0,\cdot)*V_0)(x)$, which is continuous in (t,x) (same proof as the continuity of $K_\lambda(t,\cdot)*V_0(x)$); hence, the first term of $\partial_x\Phi_T(V)$ is continuous on $]0,T[\times\mathbb{R}^N$. The continuity the second term is proved first by replacing $\partial_x V(s,\cdot)$ with $\partial_x V(s,\cdot)\mathbf{1}_{[\delta,T[}(s)$ (since this function is bounded, the continuity is obtained as for the last term of (5.2)), and then by letting $\delta\to 0$ (the convergence is uniform in $(t,x)\in[t_0,T[\times\mathbb{R}^N$ for all $t_0>0$).
A simple application of (3.16) then allows to prove that, for T small enough, Φ_T is contracting from the ball in E_T of radius R into itself, which proves the existence of a solution to (5.2) in E_T. The uniqueness of the bounded solution comes from the fact that, if T is small, Φ_T is contracting on the ball in $L^\infty(]0,T[\times\mathbb{R}^N)$ of radius R. ■

The spatial regularity of any weak solution u to (3.1) is then quite easy. Indeed, from (3.15) and the fact that u and ∇u are bounded, we see as in the proof above that u is continuous on $]0,T[\times\mathbb{R}^N$; moreover, the gradient of u satisfies (3.19) which proves, still using the same technique, that it is continuous. Since $(u,\nabla u)\in C_b(]0,T[\times\mathbb{R}^N)$, these equations (3.19) can be written in the form of (5.2) (with G taking into account u); hence, Proposition 5.2 says that the second spatial derivative of u is continuous on $]0,T[\times\mathbb{R}^N$ and bounded far from $t=0$. We can also write an integral equation satisfied by this second derivative, provided that we begin at an initial time $t_0>0$ instead of 0; this equation is of the kind (5.2). An induction process, using Proposition 5.2 on the successive equations satisfied by the spatial derivatives of u, then proves that (3.6) holds for spatial derivatives (all the regularities and bounds we obtain are local in time, but since the time span on which they hold is controlled, we also obtain global bounds).
To prove that u is differentiable w.r.t. t, we first notice that, if $\varphi\in C_b^2(\mathbb{R}^N)$, then $t\mapsto K_\lambda(t,\cdot)*\varphi(x)$ is derivable and $\frac{d}{dt}(K_\lambda(t,\cdot)*\varphi(x))=-g_\lambda[K_\lambda(t,\cdot)*\varphi](x)$; this is quite obvious on (1.1) if $\varphi\in\mathcal{S}(\mathbb{R}^N)$ and can be deduced for general φ by a density argument (same technique as in the proof of Proposition 2.1). With this result, it is possible to derivate (3.15), written at an initial time $t_0>0$ and with initial data $u(t_0,\cdot)\in C_b^2(\mathbb{R}^N)$, with respect to t (to derivate the integral term, we first replace it by $\int_0^{t-\delta}$ and then

let $\delta \to 0$); this proves that u satisfies (3.7). The spatial regularity of u and (2.1) then show that u is also regular in time.
The proof of (3.8) is immediate on (3.15) (the integral term tends to 0 in $L^\infty(\mathbb{R}^N)$ as $t \to 0$, and since u_0 is bounded and uniformly continuous and $(K_\lambda(t,\cdot))_{t\to 0}$ is an approximate unit, $K_\lambda(t,\cdot) * u_0 \to u_0$ uniformly on $\mathbb{R}^N$ as $t \to 0$).

The uniqueness i) and existence iii) in Theorem 3.2 are straightforward applications of a contracting fixed point on (3.15) in the space $\{u \in L^\infty(]0,T[\times\mathbb{R}^N) \mid \nabla u \in L^\infty(]0,T[\times\mathbb{R}^N)^N\}$ (the uniqueness is first local in time, and can then be extented to any time interval in the same way global uniqueness for ODEs is proved).

References

[1] Alvarez O. and Tourin A., *Viscosity solutions of nonlinear integro-differential equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **13** (1996), no. 3, 293–317.

[2] Amadori A.L., *Nonlinear integro-differential evolution problems arising in option pricing: a viscosity solutions approach*, Differential Integral Equations, **16** (2003), no. 7, 787–811.

[3] Barles G., Solutions de viscosité des équations de Hamilton-Jacobi, Mathématiques et applications, Springer-Verlag, 1994.

[4] Biler P., Funaki T., Woyczynski W., *Fractal Burgers Equations*, J. Diff. Eq., **148** (1998), 9–46.

[5] Biler P., Karch G., Woyczynski W., *Asymptotics for multifractal conservation laws*, Studia Math., **135** (1999), no. 3, 231–252.

[6] Benth F.E., Karlsen K.H., and Reikvam K., *Optimal portfolio selection with consumption and nonlinear integro-differential equations with gradient constraint: a viscosity solution approach*, Finance Stoch., **5** (2001), no. 3, 275–303.

[7] Benth F.E., Karlsen K.H., and Reikvam K., *Optimal portfolio management rules in a non-Gaussian market with durability and intertemporal substitution*, Finance Stoch., **5** (2001), no. 4, 447–467.

[8] Benth F.E., Karlsen K.H., and Reikvam K., *Portfolio optimization in a Lévy market with intertemporal substitution and transaction costs*, Stoch. Stoch. Rep., **74** (2002), no. 3-4, 517–569.

[9] Clavin P., *Instabilities and nonlinear patterns of overdriven detonations in gases*, H. Berestycki and Y. Pomeau (eds.), Nonlinear PDE's in Condensed Matter and Reactive Flows, Kluwer (2002), 49-97.

[10] Clarke F.H., Ledyaev Yu.S., Stern R.J., Wolenski P.R., Nonsmooth analysis and control theory, Graduate texts in mathematics, **178**, Springer, 1997.

[11] Córdoba A., Córdoba D., *A maximum principle applied to quasi-geostrophic equations*, Comm. Math. Phys., **249** (2004), no. 3, 511–528.

[12] Crandall M.G., Ishii H., Lions P.-L., *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.), **27** (1992), no. 1, 1–67.

[13] Droniou J., *Vanishing non-local regularization of a scalar conservation law*, Electron. J. Differential Equations, **2003** (2003), no. 117, 1-20.

[14] Droniou J., *Etude théorique et numérique d'équations aux dérivées partielles elliptiques, paraboliques et non-locales*, Mémoire d'Habilitation à Diriger les Recherches, Université Montpellier II, France. Available at http://www-gm3.univ-mrs.fr/~droniou/travaux-en.html.

[15] Droniou J., Gallouët T., Vovelle J., *Global solution and smoothing effect for a non-local regularization of an hyperbolic equation*, J. Evol. Equ. **3** (2003), no. 3, 499-521.

[16] Friedman A., Partial differential equations of parabolic type, Prentice-Hall, Inc., Englewood Cliffs, N.J. 1964.

[17] Garroni M.G. and Menaldi J.L., Green functions for second order parabolic integro-differential problems, Longman Scientific and Technical, Burnt Mill, Harlow, 1992

[18] Imbert C., *A non-local reguralization of first order Hamilton-Jacobi equations*, to appear in Journal of Differential Equations.

[19] Jakobsen E.R. and Karlsen K.H., *Continuous dependence estimates for viscosity solutions of integro-fs*, Preprint.

[20] Jakobsen E.R. and Karlsen K.H., *A "maximum principle for semicontinuous functions" applicable to integro-partial differential equations*, Preprint.

[21] Krushkov S.N., *First Order quasilinear equations with several space variables*, Math. USSR. Sb., **10** (1970), 217-243.

[22] Kuznecov N. N., *The accuracy of certain approximate methods for the computation of weak solutions of a first order quasilinear equation*, Ž. Vyčisl. Mat. i Mat. Fiz., **16** (1976), pp. 1489–1502, 1627.

[23] Sayah A., *Équations d'Hamilton-Jacobi du premier ordre avec termes intégro-différentiels. I. Unicité des solutions de viscosité, II. Existence de solutions de viscosité*, Comm. Partial Differential Equations, **16** (6-7), 1057–1093, 1991.

[24] Soner H.M., *Optimal control with state-space constraint. II*, SIAM J. Control Optim. **24** (1986), no. 6, 1110–1122.

[25] Soner H.M. *Optimal control of jump-Markov processes and viscosity solutions*, Stochastic differential systems, stochastic control theory and applications (Minneapolis, Minn., 1986), 501–511, IMA Vol. Math. Appl., **10**, Springer, New York, 1988.

[26] Taylor M.E., Partial Differential Equations III (nonlinear equations), Applied Mathematical Sciences, **117**, Springer-Verlag, New York, 1997.

[27] Woyczyński W.A., *Lévy processes in the physical sciences*, Lévy processes, 241-266, Birkhäuser Boston, Boston, MA, 2001.