Potential Analysis **17:** 181–203, 2002.

Non-coercive Linear Elliptic Problems

JÉRÔME DRONIOU
Université de Provence, CMI, Technopôle de Château Gombert, 39 rue F. Joliot Curie, 13453 Marseille Cedex 13, France (e-mail: droniou@cmi.univ-mrs.fr)

(Received: 14 July 2000; accepted: 13 July 2001)

Abstract. We study here some linear elliptic partial differential equations (with Dirichlet, Fourier or mixed boundary conditions), to which convection terms (first order perturbations) are added that entail the loss of the classical coercivity property. We prove the existence, uniqueness and regularity results for the solutions to these problems.

Mathematics Subject Classification (2000): 35J25.

Key words: linear elliptic PDEs, coercivity, convection terms, duality solution.

1. Introduction

1.1. NOTATION

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geqslant 2$) with a Lipschitz continuous boundary. We denote by $\mathbf{n}$ the unit normal to $\partial\Omega$ outward to Ω and by σ the measure on $\partial\Omega$.

$x \cdot y$ denotes the usual Euclidean scalar product of two vectors $(x, y) \in \mathbb{R}^N$; $|\cdot|$ is the associated Euclidean norm.

When E is a measurable subset of $\mathbb{R}^N$, $|E|$ denotes the Lebesgue measure of E.

For $q \in [1, \infty]$, q' denotes the conjugate exponent of q (that is to say $1/q + 1/q' = 1$). The space $(L^q(\Omega))^N$ is endowed with the norm $||F||_{(L^q(\Omega))^N} = ||\,|F|\,||_{L^q(\Omega)}$; $B(q, R)$ denotes the closed ball in $(L^q(\Omega))^N$ of center 0 and radius R.

If Γ is a measurable subset of $\partial\Omega$, $W^{1,q}_{\Gamma}(\Omega)$ is the space of all functions in $W^{1,q}(\Omega)$ (the usual Sobolev space) the trace of which is null on Γ; it is endowed with the same norm as $W^{1,q}(\Omega)$, that is to say $||v||_{W^{1,q}(\Omega)} = ||v||_{L^q(\Omega)} + ||\,|\nabla v|\,||_{L^q(\Omega)}$. When $q = 2$, we denote as usual $W^{1,2} = H^1$.

We take, $N_* = N$ when $N \geqslant 3$, and, $N_* \in]2, \infty[$ when $N = 2$.

1.2. THE EQUATIONS

The kinds of equations we will study are:

$$\begin{aligned} &-\mathrm{div}(A\nabla\mathcal{U}) - \mathrm{div}(\mathbf{v}\mathcal{U}) + b\mathcal{U} = \mathcal{L} && \text{in } \Omega, \\ &\mathcal{U} = \mathcal{U}_d && \text{on } \Gamma_d, \\ &A\nabla\mathcal{U}\cdot\mathbf{n} + (\lambda + \mathbf{v}\cdot\mathbf{n})\mathcal{U} = \mathcal{U}_f && \text{on } \Gamma_f \end{aligned} \tag{1}$$

and

$$\begin{cases} -\mathrm{div}(A^{\mathrm{T}}\nabla\mathcal{V})+\mathbf{v}\cdot\nabla\mathcal{V}+b\mathcal{V}=\mathcal{L} & \text{in } \Omega, \\ \mathcal{V}=\mathcal{V}_d & \text{on } \Gamma_d, \\ A^{\mathrm{T}}\nabla\mathcal{V}\cdot\mathbf{n}+\lambda\mathcal{V}=\mathcal{V}_f & \text{on } \Gamma_f \end{cases} \tag{2}$$

(where Γ_d and Γ_f are measurable subsets of $\partial\Omega$, the union of which is $\partial\Omega$ and such that $\sigma(\Gamma_d\cap\Gamma_f)=0$).

In fact, we will only study the variational (or weak) formulations of these equations; using functions $\widetilde{\mathcal{U}}$ and $\widetilde{\mathcal{V}}$ the trace on $\partial\Omega$ of which are $\mathcal{U}_d$ and $\mathcal{V}_d$, searching for weak solutions of (1) or (2) comes down to searching for solutions of

$$\begin{cases} u\in H^1_{\Gamma_d}(\Omega), \\ \displaystyle\int_\Omega A\nabla u\cdot\nabla\varphi+\int_\Omega u\mathbf{v}\cdot\nabla\varphi+\int_\Omega bu\varphi+\int_{\Gamma_f}\lambda u\varphi\,\mathrm{d}\sigma \\ \quad =\langle L,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}, \quad \forall\varphi\in H^1_{\Gamma_d}(\Omega) \end{cases} \tag{3}$$

or

$$\begin{cases} v\in H^1_{\Gamma_d}(\Omega), \\ \displaystyle\int_\Omega A^{\mathrm{T}}\nabla v\cdot\nabla\varphi+\int_\Omega \varphi\mathbf{v}\cdot\nabla v+\int_\Omega bv\varphi+\int_{\Gamma_f}\lambda v\varphi\,\mathrm{d}\sigma \\ \quad =\langle L,\varphi\rangle_{(H^1_{\Gamma_d}(\Omega))',H^1_{\Gamma_d}(\Omega)}, \quad \forall\varphi\in H^1_{\Gamma_d}(\Omega) \end{cases} \tag{4}$$

(with $u=\mathcal{U}-\widetilde{\mathcal{U}}$, $v=\mathcal{V}-\widetilde{\mathcal{V}}$ and L which takes into account $\mathcal{L}$ and $\widetilde{\mathcal{U}}$ and $\mathcal{U}_f$ or $\widetilde{\mathcal{V}}$ and $\mathcal{V}_f$).

In order that all the terms in (3) and (4) be defined, the minimal hypotheses on the data are, thanks to the Sobolev injections: A: $\Omega\to M_N(\mathbb{R})$ is a matrix-valued essentially bounded measurable function, $b\in L^{\frac{N_*}{2}}(\Omega)$, $\lambda\in L^{N_*-1}(\partial\Omega)$ and $\mathbf{v}\in(L^{N_*}(\Omega))^N$.

The classical framework of study for linear elliptic problems is the Lax–Milgram Theorem, which demands the coercivity of the bilinear form appearing in (3) or (4), i.e., additional hypotheses on the data.

The main coercivity hypothesis is on A, to ensure that the principal part of the operator is elliptic (see hypothesis (9)).

In order that the lower order terms do not cause the loss of this coercivity, it is usual to add then hypotheses on $\mathbf{v}$, b, and λ. For the pure Dirichlet condition ($\Gamma_f=\emptyset$), this can be

$$-\frac{1}{2}\mathrm{div}(\mathbf{v})+b\geqslant c \quad \text{in } \mathcal{D}'(\Omega),$$

with c "small enough" in $L^{\frac{N_*}{2}}(\Omega)$ (in general, c is taken equal to 0) – note that this condition adds hypothesis on the *regularity* of $\mathbf{v}$ (when this inequality is satisfied, $\mathrm{div}(\mathbf{v})$ must be a Radon measure on Ω).

In the case of Fourier or mixed boundary conditions, to clearly express these additional hypotheses, we need more regularity on $\mathbf{v}$ (to give a sense to $\mathbf{v}\cdot\mathbf{n}$).

Moreover, in these cases, when we want to obtain regularity results, the minimal regularity on $\mathbf{v}$ seemed to be the Lipschitz continuity (because of the many integrations by parts we have then to do; see [6]).

Asking for the principal part ($-\text{div}(A\nabla u)$ or $-\text{div}(A^{\mathrm{T}}\nabla v)$) to be coercive is quite natural when we search for solutions in $H^1(\Omega)$. One could wonder if the additional hypotheses on the lower order terms $\text{div}(\mathbf{v}u)$ (or $\mathbf{v}\cdot\nabla u$), bu and λu are really necessary; we will see below that we cannot avoid some hypotheses on the zero-order terms bu and λu. Concerning the first-order terms, work has already been done to get rid of the coercivity hypothesis on the convection term when it is in conservative form.

In [3], the author proves an existence result and studies some qualitative properties for entropy solutions of

$$\begin{array}{ll} -\text{div}(a(x,u,\nabla u)) = f - \text{div}(F + \Phi(u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \tag{5}$$

where $\text{div}(a(x,u,\nabla u))$ is a Leray–Lions operator in divergence form acting on $W_0^{1,p}(\Omega)$ $(1 - 2/N < p < N)$, $f \in L^1(\Omega)$, $F \in (L^{p'}(\Omega))^N$ and Φ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^N$; due to the lack of growth properties on Φ, it is crucial in (5) to consider pure homogeneous Dirichlet boundary conditions and a Φ not depending on $x \in \Omega$.

In [2], the authors study the existence and uniqueness of renormalized solutions to

$$\begin{array}{ll} \lambda u - \text{div}(a(x,\nabla u) + \Phi(x,u)) = f & \text{in } \Omega, \\ (a(x,\nabla u) + \Phi(x,u))\cdot\mathbf{n} = 0 & \text{on } \Gamma_f, \\ u = 0 & \text{on } \Gamma_d, \end{array} \tag{6}$$

where λ is a non-negative real number, $\text{div}(a(x,\nabla u))$ is a Leray–Lions operator in divergence form – note the independence of a with respect to u – acting on $W^{1,p}(\Omega)$, $f \in L^1(\Omega)$ and Φ is a Caratheodory function with growth properties; the problem is either pure Dirichlet ($\Gamma_f = \emptyset$) or mixed ($\Gamma_f \neq \emptyset$ but $\sigma(\Gamma_d) > 0$).

We prove, in Section 2, the existence and uniqueness results for (3) and (4), with no coercivity hypothesis on the convection term. These results are not consequences of [3] or [2], because the natural space of entropy or renormalized solutions is not the usual Sobolev space $H^1(\Omega)$.

In Section 3, we will see that the regularity results we already have in the coercive case, where the right-hand side L is more regular (see [9] and [6]), are still true with general convection terms. Under stronger hypotheses ($\mathbf{v} \in (L^\infty(\Omega))^N$ and $L \in L^\infty(\Omega)$), the existence and regularity results appear in [7].

We then briefly describe, in Section 4, how the regularity results of Section 3 can be transformed in the existence and uniqueness results with measures as data (as in [9] or [6]).

1.3. THE ZERO-ORDER TERMS

We cannot, in general, solve problems (3) and (4) for any $b \in L^{N_*/2}(\Omega)$ and $\lambda \in L^{N_*-1}(\partial\Omega)$. This is due to the existence of an eigenvalue for the Laplace operator.

Consider pure Dirichlet boundary conditions and take e an eigenvector of $-\Delta$ on $H_0^1(\Omega)$, that is to say, $e \in H_0^1(\Omega)\backslash\{0\}$ such that $-\Delta e = le$ for $l \in \mathbb{R}$ (in fact, we have then $l > 0$).

Take now $b \in \mathbb{R}$ and suppose there exists a solution $u \in H_0^1(\Omega)$ of $-\Delta u + bu = e$; that is, for all $\varphi \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega bu\varphi = \int_\Omega e\varphi.$$

With $\varphi = e$, we get

$$(l + b) \int_\Omega ue = \int_\Omega e^2.$$

Since $e \neq 0$, this last equation cannot be satisfied for $b = -l$; thus, there is no solution $u \in H_0^1(\Omega)$ of $-\Delta u - lu = e$.

The same kind of reasoning can be done in the mixed case, and this shows that we cannot avoid additional hypotheses on b and λ (i.e., we cannot only suppose integrability hypotheses on these data).

In (3), we have considered convection terms only in conservative form; in (4), we have considered convection terms only in non-conservative form. A natural question is the following: can we consider, in the same equation, convection terms both in conservative and non-conservative form? That is to say, can we solve

$$\begin{array}{ll} -\mathrm{div}(A\nabla u) - \mathrm{div}(\mathbf{v}u) + \mathbf{w} \cdot \nabla u + bu = L & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_d, \\ A\nabla u \cdot \mathbf{n} + (\lambda + \mathbf{v} \cdot \mathbf{n})u = 0 & \text{on } \Gamma_f, \end{array} \tag{7}$$

in the same way as we solve (3) and (4) (i.e., without an additional hypothesis on the convection terms)?

The answer is no and is due to the same objection as before. Indeed, take $\mathbf{v}$ as a regular vector-valued function; since, for $u \in H_0^1(\Omega)$, we have $\mathrm{div}(\mathbf{v}u) - \mathbf{v} \cdot \nabla u = u\mathrm{div}(\mathbf{v})$, a solution in $H_0^1(\Omega)$ of $-\Delta u - \mathrm{div}(\mathbf{v}u) + \mathbf{v} \cdot \nabla u = L$ (that is, problem (7) in the case of pure Dirichlet boundary conditions, with $A = \mathrm{Id}$, $b = 0$ and $\mathbf{w} = \mathbf{v}$) would be a solution to $-\Delta u - (\mathrm{div}(\mathbf{v}))u = L$; by taking a regular vector-valued function $\mathbf{v}$ such that $\mathrm{div}(\mathbf{v}) = l$, the preceding reasoning proves that, in general, this last problem has no solution.

Thus, (7) is not solvable without additional hypotheses on the first-order terms.

Problems (3) and (4) seem thus to be the most general problems we can solve, when we add no structural hypothesis on the first-order terms.

1.4. HYPOTHESES

We make the following hypotheses on the data.

$$\begin{aligned}&\Gamma_d \text{ and } \Gamma_f \text{ are measurable subsets of } \partial\Omega \text{ such that}\\ &\quad \sigma(\Gamma_d \cap \Gamma_f) = 0 \text{ and } \partial\Omega = \Gamma_d \cup \Gamma_f,\end{aligned} \tag{8}$$

$$\begin{aligned}&A\colon\ \Omega \to M_N(\mathbb{R}) \quad \text{is a measurable matrix-valued function which satisfies:}\\ &\exists \alpha_A > 0 \text{ s.t. } A(x)\xi\cdot\xi \geqslant \alpha_A|\xi|^2 \quad \text{for a.e. } x \in \Omega, \text{ for all } \xi \in \mathbb{R}^N,\\ &\exists \Lambda_A > 0 \text{ s.t. } ||A(x)|| \leqslant \Lambda_A \quad \text{for a.e. } x \in \Omega\end{aligned} \tag{9}$$

(where, for $M \in M_N(\mathbb{R})$, $||M|| := \sup\{|M\xi|\,,\ \xi \in \mathbb{R}^N\,,\ |\xi| = 1\}$),

$$b \in L^{N_*/2}(\Omega), \quad b \geqslant 0 \text{ a.e. on } \Omega, \tag{10}$$

$$\lambda \in L^{N_*-1}(\partial\Omega), \quad \lambda \geqslant 0\ \sigma\text{-a.e. on } \partial\Omega, \tag{11}$$

$$\mathbf{v} \in (L^{N_*}(\Omega))^N, \tag{12}$$

$$L \in (H^1_{\Gamma_d}(\Omega))' \tag{13}$$

(recall that $N_* = N$ when $N \geqslant 3$ and that $N_* \in]2, \infty[$ when $N = 2$).

The non-convection parts of equations (3) and (4) are supposed to be coercive, that is to say:

$$\begin{aligned}&\exists b_0 > 0, \exists E \subset \Omega \text{ such that } b \geqslant b_0 \text{ on } E,\\ &\exists \lambda_0 > 0, \exists S \subset \Gamma_f \text{ such that } \lambda \geqslant \lambda_0 \text{ on } S \text{ and either}\\ &\sigma(\Gamma_d) > 0 \text{ or } |E| > 0 \text{ or } \sigma(S) > 0.\end{aligned} \tag{14}$$

The set of variables that give the coercivity of the principal part of the operators in (3) or (4) is denoted by $\mathcal{B} = (\Omega, \alpha_A, \Gamma_d, b_0, E, \lambda_0, S)$.

REMARK 1.1. It is then well known that, under hypotheses (8)–(11) and (14), for all $q \in [1, 2]$, there exists $\mathcal{K}(q, \mathcal{B}) > 0$ such that, for all $\varphi \in H^1_{\Gamma_d}(\Omega)$,

$$\mathcal{K}(q, \mathcal{B})||\varphi||^2_{H^1(\Omega)} \leqslant \alpha_A \int_\Omega |\nabla\varphi|^2 + \left(b_0 \int_E |\varphi|^q + \lambda_0 \int_S |\varphi|^q \,\mathrm{d}\sigma\right)^{2/q}.$$

Denoting by $C_S(\Omega, N_*)$ the norm of the Sobolev injection $H^1(\Omega) \hookrightarrow L^{\frac{2N_*}{N_*-2}}(\Omega)$ (see [1]), we also take

$$\chi \in \left[0, \frac{\mathcal{K}(2, \mathcal{B})}{C_S(\Omega, N_*)}\right[. \tag{15}$$

REMARK 1.2. When χ satisfies (15), we have, for all $\mathbf{w} \in B(N_*, \chi)$ and all $\varphi \in H^1_{\Gamma_d}(\Omega)$, that

$$\begin{aligned}\int_\Omega A\nabla\varphi\cdot\nabla\varphi + \int_\Omega \varphi\mathbf{w}\cdot\nabla\varphi + \int_\Omega b\varphi^2 + \int_{\Gamma_f} \lambda\varphi^2\,\mathrm{d}\sigma \\ \geqslant \mathcal{K}(2,\mathcal{B})||\varphi||^2_{H^1(\Omega)} - ||\,|\mathbf{w}|\,||_{L^{N_*}(\Omega)}||\varphi||_{L^{2N_*/(N_*-2)}(\Omega)}||\,|\nabla\varphi|\,||_{L^2(\Omega)} \\ \geqslant (\mathcal{K}(2,\mathcal{B}) - \chi C_S(\Omega, N_*))||\varphi||^2_{H^1(\Omega)},\end{aligned}$$

with $\mathcal{K}(2,\mathcal{B}) - \chi C_S(\Omega, N_*) > 0$ (this exactly means that the bilinear form $(\varphi,\psi) \to \int_\Omega A\nabla\varphi\cdot\nabla\psi + \int_\Omega \varphi\mathbf{w}\cdot\nabla\psi + \int_\Omega b\varphi\psi + \int_{\Gamma_f}\lambda\varphi\psi\,\mathrm{d}\sigma$ is coercive on $H^1_{\Gamma_d}(\Omega)$).

If $\mathbf{v} \in B(N_*, \chi)$ with χ satisfying (15), by the Lax–Milgram Theorem, (3) and (4) have thus unique solutions; our aim is to prove that we do not need such a hypothesis on $\mathbf{v}$ to have the existence and uniqueness results for these problems.

When $\mathbf{v}$ only satisfies (12), problems (3) and (4) are in general non-coercive not only in the sense of the Lax–Milgram Theorem (the classical tool for linear elliptic problems), but also in the sense of the Leray–Lions Theorem (the classical tool for nonlinear elliptic problems). Indeed, consider the pure Dirichlet boundary conditions with $b = \lambda = 0$ (for the sake of simplicity) and take $\mathbf{w}$ as a regular function such that $\mathrm{div}(\mathbf{w}) \neq 0$; we can find $u \in H^1_0(\Omega)$ such that

$$\int_\Omega u\mathbf{w}\cdot\nabla u = \frac{1}{2}\int_\Omega \mathbf{w}\cdot\nabla(u^2) \neq 0$$

(take $u \in \mathcal{C}^\infty_c(\Omega)\backslash\{0\}$ such that $\mathrm{supp}(u) \subset \{x \in \Omega \mid \mathrm{div}(\mathbf{w})(x) < 0\}$ or $\mathrm{supp}(u) \subset \{x \in \Omega \mid \mathrm{div}(\mathbf{w})(x) > 0\}$); let then

$$s = -\frac{\int_\Omega A\nabla u\cdot\nabla u}{\int_\Omega u\mathbf{w}\cdot\nabla u} \quad \text{and} \quad \mathbf{v} = s\mathbf{w}.$$

The sequence $(u_n)_{n\geqslant 1} = (nu)_{n\geqslant 1} \in H^1_0(\Omega)$ satisfies $||u_n||_{H^1_0(\Omega)} \to \infty$ as $n \to \infty$ and

$$\int_\Omega A\nabla u_n\cdot\nabla u_n + \int_\Omega u_n\mathbf{v}\cdot\nabla u_n = 0 \quad \text{for all } n \geqslant 1,$$

which means that the operator in (3) or (4) is not coercive in the sense of Leray–Lions (see [8]).

Also note that, when A satisfies (9), A^{T} also satisfies (9); thus, in (4), we could replace A^{T} by A. We have written (4) with A^{T} so that the duality between (3) and (4) clearly appears.

2. Existence and Uniqueness Results

2.1. THE MAIN RESULT

THEOREM 2.1. *Under hypotheses* (8)–(14)*, there exists a unique solution* u *to* (3) *and a unique solution* v *to* (4)*. Moreover, if* $r > N$ *and* $\Lambda \geqslant 0$ *are such that* $\mathbf{v} \in B(N_*, \chi) + B(r, \Lambda)$*, with* χ *satisfying* (15)*, and if* Λ_L *is an upper bound of* $||L||_{(H^1_{\Gamma_d}(\Omega))'}$*, there exists* C *only depending on* $(N_*, \mathcal{B}, \chi, r, \Lambda, \Lambda_L)$ *such that* $||u||_{H^1(\Omega)} \leqslant C$ *and* $||v||_{H^1(\Omega)} \leqslant C$.

Note that, for all $\mathbf{v} \in (L^{N_*}(\Omega))^N$ and all $\eta > 0$, there exists $\Lambda > 0$ such that $\mathbf{v} \in B(N_*, \eta) + B(\infty, \Lambda)$; however, this Λ does not only depend on the norm of $\mathbf{v}$ in $(L^{N_*}(\Omega))^N$. On the other hand, if $\mathbf{v}$ is in a compact subset K of $(L^{N_*}(\Omega))^N$, for example, we can choose Λ only depending on K and η.

We can also remark that, in the pure Dirichlet case ($\Gamma_f = \emptyset$), the Lipschitz continuity hypothesis on the boundary of Ω is useless in Theorem 2.1.

Proof of Theorem 2.1. The proof is performed in several steps. The main tool to obtain the existence and estimates of the solutions of (3) and (4) is the Leray–Schauder Topological Degree (see [5]).

The first three steps are devoted to prove an existence result for (3). This existence result is then used in the fourth and fifth steps to prove an *a priori* estimate on the solution of (4) that lead to an existence result for (4). Using the linearity of these equations and a duality argument, we prove, in the last step, the uniqueness results.

We will simultaneously obtain the existence of solutions to (3) and (4) and the estimates given in the theorem; thus, we take from now on $r > N$, $\Lambda \geqslant 0$ and χ satisfying (15), and we suppose that $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$ with $(\mathbf{v}_0, \mathbf{v}_1) \in (L^{N_*}(\Omega))^N \times (L^r(\Omega))^N$, $||\,|\mathbf{v}_0|\,||_{L^{N_*}(\Omega)} \leqslant \chi$ and $||\,|\mathbf{v}_1|\,||_{L^r(\Omega)} \leqslant \Lambda$. We see that the bound in $H^1(\Omega)$ on the solutions we obtain only depends on $(N_*, \mathcal{B}, \chi, r, \Lambda, \Lambda_L)$.

Step 1: a compact application to (3).

For all $\overline{u} \in H^1_{\Gamma_d}(\Omega)$, since $\overline{u}\mathbf{v} \in (L^2(\Omega))^N$ (because of the Sobolev injection $H^1(\Omega) \hookrightarrow L^{2N_*/(N_*-2)}(\Omega)$), there exists a unique $u = \mathcal{F}(\overline{u})$ solution to

$$\begin{aligned} & u \in H^1_{\Gamma_d}(\Omega), \\ & \int_\Omega A\nabla u \cdot \nabla\varphi + \int_\Omega bu\varphi + \int_{\Gamma_f} \lambda u\varphi \,\mathrm{d}\sigma \\ & \quad = \langle L, \varphi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)} - \int_\Omega \overline{u}\mathbf{v} \cdot \nabla\varphi, \quad \forall\varphi \in H^1_{\Gamma_d}(\Omega). \end{aligned} \tag{16}$$

This defines an application $\mathcal{F}\colon H^1_{\Gamma_d}(\Omega) \to H^1_{\Gamma_d}(\Omega)$.

It is quite easy to see that $\mathcal{F}$ is continuous; indeed, if $\overline{u}_n \to \overline{u}$ in $H^1_{\Gamma_d}(\Omega)$ as $n \to \infty$, then $\overline{u}_n\mathbf{v} \to \overline{u}\mathbf{v}$ in $(L^2(\Omega))^N$, so that $\mathcal{F}(\overline{u}_n) \to \mathcal{F}(\overline{u})$ in $H^1(\Omega)$.

Suppose that $(\overline{u}_n)_{n\geqslant 1}$ is a bounded sequence of $H^1_{\Gamma_d}(\Omega)$. There exists then $\overline{u} \in H^1_{\Gamma_d}(\Omega)$ such that, up to a subsequence, $\overline{u}_n \to \overline{u}$ a.e. on Ω and is bounded in $L^{2N_*/(N_*-2)}(\Omega)$; applying Lemma A.1, we get $\overline{u}_n\mathbf{v} \to \overline{u}\mathbf{v}$ in $(L^2(\Omega))^N$, which implies $\mathcal{F}(\overline{u}_n) \to \mathcal{F}(\overline{u})$ in $H^1(\Omega)$. $\mathcal{F}$ is thus a compact operator.

A fixed point of $\mathcal{F}$ is a solution to (3). To prove, using the Leray–Schauder Topological Degree, that $\mathcal{F}$ has a fixed point, we have to find $R > 0$ such that, for all $t \in [0, 1]$, there exists no solution of $u - t\mathcal{F}(u) = 0$ satisfying $||u||_{H^1(\Omega)} = R$. This is the aim of steps two and three.

Take $t \in [0, 1]$ and suppose that u satisfies $u = t\mathcal{F}(u)$; we have then

$$\begin{aligned} & u \in H^1_{\Gamma_d}(\Omega), \\ & \int_\Omega A\nabla u \cdot \nabla\varphi + \int_\Omega bu\varphi + \int_{\Gamma_f} \lambda u\varphi \,\mathrm{d}\sigma \\ & \quad = t\langle L, \varphi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)} - t\int_\Omega u\mathbf{v} \cdot \nabla\varphi, \quad \forall\varphi \in H^1_{\Gamma_d}(\Omega). \end{aligned} \tag{17}$$

Notice that the equation in (17) can also be written as

$$\begin{aligned} & \int_\Omega A\nabla u \cdot \nabla\varphi + t\int_\Omega u\mathbf{v}_0 \cdot \nabla\varphi + \int_\Omega bu\varphi + \int_{\Gamma_f} \lambda u\varphi \,\mathrm{d}\sigma \\ & \quad = t\langle L, \varphi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)} - t\int_\Omega u\mathbf{v}_1 \cdot \nabla\varphi. \end{aligned} \tag{18}$$

Step 2: using the ideas of [4], we prove an estimate on $\ln(1 + |u|)$.

Define, for $k \geqslant 0$, $T_k(s) = \min(k, \max(-s, k))$ and $r_k(s) = T_1(s - T_k(s))$. Since $b \geqslant 0$ a.e. on Ω, $\lambda \geqslant 0$ σ-a.e. on $\partial\Omega$ and $sr_k(s) \geqslant 0$ for all $s \in \mathbb{R}$, and since $\nabla(r_k(u)) = \mathbf{1}_{B_k}\nabla u$, with $\mathbf{1}_{B_k}$ the characteristic function of the set $B_k = \{x \in \Omega \mid k \leqslant |u| < k + 1\}$, we find, by putting $\varphi = r_k(u)$ in (17), that

$$\begin{aligned} & \alpha_A \int_\Omega |\nabla(r_k(u))|^2 + b_0\int_E r_k(u)u + \lambda_0\int_S r_k(u)u \,\mathrm{d}\sigma \\ & \quad \leqslant \int_\Omega A\nabla u \cdot \nabla(r_k(u)) + \int_\Omega bur_k(u) + \int_{\Gamma_f} \lambda ur_k(u)\,\mathrm{d}\sigma \\ & \quad \leqslant |\langle L, r_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}| + \int_\Omega |u||\mathbf{v}||\nabla(r_k(u))| \\ & \quad \leqslant ||L||_{(H^1_{\Gamma_d}(\Omega))'}||r_k(u)||_{H^1(\Omega)} + (k+1)||\,|\mathbf{v}|\,||_{L^2(B_k)}||\,|\nabla(r_k(u))|\,||_{L^2(\Omega)}. \end{aligned}$$

But $|r_k(s)| \leqslant 1$ so that

$$||r_k(u)||_{H^1(\Omega)} \leqslant |\Omega|^{1/2} + ||\,|\nabla(r_k(u))|\,||_{L^2(\Omega)}.$$

We obtain thus

$$\begin{aligned} & \alpha_A \int_\Omega |\nabla(r_k(u))|^2 + b_0\int_E r_k(u)u + \lambda_0\int_S r_k(u)u \,\mathrm{d}\sigma \\ & \quad \leqslant \Lambda_L|\Omega|^{1/2} + \Lambda_L||\,|\nabla(r_k(u))|\,||_{L^2(\Omega)} + \\ & \qquad + (k+1)||\,|\mathbf{v}|\,||_{L^2(B_k)}||\,|\nabla(r_k(u))|\,||_{L^2(\Omega)} \end{aligned}$$

$$\leqslant \Lambda_L|\Omega|^{1/2} + \frac{\Lambda_L^2}{\alpha_A} + \frac{\alpha_A}{4}||\,|\nabla(r_k(u))|\,||^2_{L^2(\Omega)} + \frac{\alpha_A}{4}||\,|\nabla(r_k(u))|\,||^2_{L^2(\Omega)} + \\ + (k+1)^2\frac{||\,|\mathbf{v}|\,||^2_{L^2(B_k)}}{\alpha_A},$$

that is to say,

$$\frac{\alpha_A}{2}||\,|\nabla(r_k(u))|\,||^2_{L^2(\Omega)} + b_0\int_E r_k(u)u + \lambda_0\int_S r_k(u)u\,\mathrm{d}\sigma \\ \leqslant \Lambda_L|\Omega|^{1/2} + \frac{\Lambda_L^2}{\alpha_A} + (k+1)^2\frac{||\,|\mathbf{v}|\,||^2_{L^2(B_k)}}{\alpha_A}. \tag{19}$$

With $k = 0$, since $sr_0(s) = |s|$ as soon as $|s| \geqslant 1$, (19) gives

$$b_0\int_E \ln(1+|u|) + \lambda_0\int_S \ln(1+|u|)\,\mathrm{d}\sigma \\ \leqslant b_0\int_E |u| + \lambda_0\int_S |u|\,\mathrm{d}\sigma \\ \leqslant b_0\int_{E\cap\{|u|\geqslant 1\}} r_0(u)u + \lambda_0\int_{S\cap\{|u|\geqslant 1\}} r_0(u)u\,\mathrm{d}\sigma + b_0\int_{E\cap\{|u|\leqslant 1\}}|u| + \\ + \lambda_0\int_{S\cap\{|u|\leqslant 1\}}|u|\,\mathrm{d}\sigma \\ \leqslant \Lambda_L|\Omega|^{1/2} + \frac{\Lambda_L^2}{\alpha_A} + \frac{||\,|\mathbf{v}|\,||^2_{L^2(\Omega)}}{\alpha_A} + b_0|E| + \lambda_0\sigma(S) \\ \leqslant \Lambda_L|\Omega|^{1/2} + \frac{\Lambda_L^2}{\alpha_A} + \frac{2}{\alpha_A}(||\,|\mathbf{v}_0|\,||^2_{L^2(\Omega)} + ||\,|\mathbf{v}_1|\,||^2_{L^2(\Omega)}) + b_0|E| + \lambda_0\sigma(S) \\ \leqslant \Lambda_L|\Omega|^{1/2} + \frac{\Lambda_L^2}{\alpha_A} + \frac{2}{\alpha_A}\big(|\Omega|^{1-2/N_*}\chi^2 + |\Omega|^{1-\frac{2}{r}}\Lambda^2\big) + b_0|E| + \lambda_0\sigma(S) \\ \leqslant C_1 \tag{20}$$

(recall that $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$ with $||\,|\mathbf{v}_0|\,||_{L^{N_*}(\Omega)} \leqslant \chi$ and $||\,|\mathbf{v}_1|\,||_{L^r(\Omega)} \leqslant \Lambda$), with C_1 only depending on $(N_*, \mathcal{B}, \chi, r, \Lambda, \Lambda_L)$.

Since $(B_k)_{k\in\mathbb{N}}$ is a partition of Ω, and since $|u| \geqslant k$ on B_k, we find, using once again (19), that

$$||\,|\nabla(\ln(1+|u|))|\,||^2_{L^2(\Omega)} \\ = \int_\Omega \frac{|\nabla u|^2}{(1+|u|)^2} \\ = \sum_{k=0}^{\infty}\int_{B_k}\frac{|\nabla u|^2}{(1+|u|)^2} \\ \leqslant \sum_{k=0}^{\infty}\int_\Omega \frac{|\nabla(r_k(u))|^2}{(1+k)^2}$$

$$
\begin{aligned}
&\leqslant \frac{2}{\alpha_A}\left(\Lambda_L|\Omega|^{1/2}+\frac{\Lambda_L^2}{\alpha_A}\right)\sum_{k=0}^{\infty}\frac{1}{(1+k)^2}+\frac{2}{\alpha_A^2}\sum_{k=0}^{\infty}\int_{B_k}|\mathbf{v}|^2\\
&\leqslant \frac{2}{\alpha_A}\frac{\pi^2}{6}\left(\Lambda_L|\Omega|^{1/2}+\frac{\Lambda_L^2}{\alpha_A}\right)+\frac{2||\,|\mathbf{v}_0|+|\mathbf{v}_1|\,||^2_{L^2(\Omega)}}{\alpha_A^2}\\
&\leqslant \frac{2}{\alpha_A}\frac{\pi^2}{6}\left(\Lambda_L|\Omega|^{1/2}+\frac{\Lambda_L^2}{\alpha_A}\right)+\frac{4|\Omega|^{1-2/N_*}\chi^2+4|\Omega|^{1-2/r}\Lambda^2}{\alpha_A^2}\\
&\leqslant C_2, \qquad (21)
\end{aligned}
$$

where C_2 only depends on $(N_*, \mathcal{B}, \chi, r, \Lambda, \Lambda_L)$.

Taking together (20) and (21) we get, thanks to Remark 1.1,

$$
\begin{aligned}
||\ln(1+|u|)||^2_{L^2(\Omega)} &\leqslant ||\ln(1+|u|)||^2_{H^1(\Omega)}\\
&\leqslant \frac{1}{\mathcal{K}(1,\mathcal{B})}(\alpha_A C_2 + C_1^2) = C_3 \qquad (22)
\end{aligned}
$$

with C_3 only depending on $(N_*, \mathcal{B}, \chi, r, \Lambda, \Lambda_L)$.

Step 3: conclusion for (3).

We prove now an $H^1(\Omega)$ estimate on the solution of (17).

Take $\varphi = S_k(u) = u - T_k(u)$ in (18). Since $S_k(u)u \geqslant (S_k(u))^2$, we have, thanks to Remark 1.2 (note that, for all $t \in [0,1]$, $t\mathbf{v}_0 \in B(N_*, \chi)$), that

$$
\begin{aligned}
&(\mathcal{K}(2,\mathcal{B}) - \chi C_S(\Omega, N_*))||S_k(u)||^2_{H^1(\Omega)}\\
&\leqslant \int_\Omega A\nabla(S_k(u))\cdot\nabla(S_k(u)) + t\int_\Omega S_k(u)\mathbf{v}_0\cdot\nabla S_k(u) +\\
&\quad + \int_\Omega b(S_k(u))^2 + \int_{\Gamma_f}\lambda(S_k(u))^2\,\mathrm{d}\sigma\\
&\leqslant \int_\Omega A\nabla u\cdot\nabla(S_k(u)) + t\int_\Omega u\mathbf{v}_0\cdot\nabla S_k(u) + \int_\Omega buS_k(u) +\\
&\quad + \int_{\Gamma_f}\lambda u S_k(u)\,\mathrm{d}\sigma + t\int_\Omega (S_k(u)-u)\mathbf{v}_0\cdot\nabla(S_k(u))\\
&\leqslant |\langle L, S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}| + \int_\Omega |u||\mathbf{v}_1||\nabla(S_k(u))| +\\
&\quad + \int_\Omega |u - S_k(u)||\mathbf{v}_0||\nabla(S_k(u))|\\
&\leqslant |\langle L, S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}| + \int_\Omega |u - S_k(u)|(|\mathbf{v}_0|+|\mathbf{v}_1|)|\nabla(S_k(u))| +\\
&\quad + \int_\Omega |S_k(u)||\mathbf{v}_1||\nabla(S_k(u))|.
\end{aligned}
$$

But $|u - S_k(u)| \leqslant k$ and $\nabla(S_k(u)) = 0$ outside $E_k = \{|u| \geqslant k\}$, so that

$$
\begin{aligned}
&(\mathcal{K}(2,\mathcal{B}) - \chi C_S(\Omega, N_*))||S_k(u)||^2_{H^1(\Omega)}\\
&\quad \leqslant |\langle L, S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}| + k|| \,|\mathbf{v}_0| + |\mathbf{v}_1|\, ||_{L^2(E_k)}||S_k(u)||_{H^1(\Omega)} +\\
&\quad\quad + ||\,|\mathbf{v}_1|\,||_{L^r(\Omega)}||S_k(u)||_{L^{2r/(r-2)}(\Omega)}||S_k(u)||_{H^1(\Omega)}\\
&\quad \leqslant |\langle L, S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}| +\\
&\quad\quad + k\big(|E_k|^{1/2-1/N_*}\chi + |E_k|^{1/2-1/r}\Lambda\big)||S_k(u)||_{H^1(\Omega)} +\\
&\quad\quad + \Lambda||S_k(u)||_{L^{2r/(r-2)}(\Omega)}||S_k(u)||_{H^1(\Omega)}.
\end{aligned} \tag{23}
$$

Since $\frac{2r}{r-2} < \frac{2N}{N-2}$, there exists $q_r > \frac{2r}{r-2}$ only depending on r and N such that $H^1(\Omega) \hookrightarrow L^{q_r}(\Omega)$; we have thus, denoting by C_4 the norm of this injection (C_4 only depends on (Ω, r) – the dependence on Ω takes into account the dependence on N) and by noticing that $S_k(u) = 0$ outside E_k,

$$
\begin{aligned}
||S_k(u)||_{L^{2r/(r-2)}(\Omega)} &\leqslant |E_k|^{(r-2)/2r-1/q_r}||S_k(u)||_{L^{q_r}(\Omega)}\\
&\leqslant C_4|E_k|^{(r-2)/2r-1/q_r}||S_k(u)||_{H^1(\Omega)},
\end{aligned}
$$

which gives, in (23),

$$
\begin{aligned}
&(\mathcal{K}(2,\mathcal{B}) - \chi C_S(\Omega, N_*))||S_k(u)||^2_{H^1(\Omega)}\\
&\quad \leqslant |\langle L, S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}| +\\
&\quad\quad + k\big(|E_k|^{1/2-1/N_*}\chi + |E_k|^{1/2-1/r}\Lambda\big)||S_k(u)||_{H^1(\Omega)} +\\
&\quad\quad + C_4\Lambda|E_k|^{(r-2)/2r-1/q_r}||S_k(u)||^2_{H^1(\Omega)}.
\end{aligned} \tag{24}
$$

By the Tchebysheff inequality and (22), we have

$$
\begin{aligned}
|E_k| &= |\{\ln(1+|u|)^2 \geqslant \ln(1+k)^2\}|\\
&\leqslant \frac{1}{(\ln(1+k))^2}||\ln(1+|u|)||^2_{L^2(\Omega)}\\
&\leqslant \frac{C_3}{(\ln(1+k))^2}.
\end{aligned}
$$

Since $\frac{r-2}{2r} - \frac{1}{q_r} > 0$, there exists k_0 only depending on C_3, C_4, Λ, r, q_r, $\mathcal{K}(2,\mathcal{B})$, χ and $C_S(\Omega, N_*)$, i.e., only depending on $(N_*, \mathcal{B}, \chi, r, \Lambda, \Lambda_L)$, such that, for all $k \geqslant k_0$, $C_4\Lambda|E_k|^{(r-2)/2r-1/q_r} \leqslant \frac{\mathcal{K}(2,\mathcal{B})-\chi C_S(\Omega,N_*)}{2}$.

We deduce then form (24) that, for all $k \geqslant k_0$,

$$
\begin{aligned}
&\frac{\mathcal{K}(2,\mathcal{B}) - \chi C_S(\Omega, N_*)}{2}||S_k(u)||_{H^1(\Omega)}\\
&\quad \leqslant \frac{|\langle L, S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}|}{||S_k(u)||_{H^1(\Omega)}} + k\big(|E_k|^{1/2-1/N_*}\chi + |E_k|^{1/2-1/r}\Lambda\big)
\end{aligned} \tag{25}
$$

(we have not simplified so far, because this inequality will be useful in the proof of Proposition 3.1).

Taking $k = k_0$, and, since $E_k \subset \Omega$, we get

$$\begin{aligned} &||S_{k_0}(u)||_{H^1(\Omega)} \\ &\leqslant \frac{2}{\mathcal{K}(2, \mathcal{B}) - \chi C_S(\Omega, N_*)}\big(\Lambda_L + k_0\big(|\Omega|^{1/2-1/N_*}\chi + |\Omega|^{1/2-1/r}\Lambda\big)\big) \\ &\leqslant C_5 \end{aligned} \tag{26}$$

with C_5 only depending on $(N_*, \mathcal{B}, \chi, r, \Lambda, \Lambda_L)$.

Take now $\varphi = T_{k_0}(u)$ in (17). Since $uT_{k_0}(u) \geqslant (T_{k_0}(u))^2$ and

$$\nabla(T_{k_0}(u)) = \mathbf{1}_{\{|u|\leqslant k_0\}}\nabla u,$$

we have, by Remark 1.1,

$$\begin{aligned} &\mathcal{K}(2, \mathcal{B})||T_{k_0}(u)||^2_{H^1(\Omega)} \\ &\leqslant \Lambda_L||T_{k_0}(u)||_{H^1(\Omega)} + \int_\Omega |u||\mathbf{v}||\nabla(T_{k_0}(u))| \\ &\leqslant \Lambda_L||T_{k_0}(u)||_{H^1(\Omega)} + k_0||\,|\mathbf{v}_0| + |\mathbf{v}_1|\,||_{L^2(\Omega)}||T_{k_0}(u))||_{H^1(\Omega)}, \end{aligned}$$

that is to say,

$$||T_{k_0}(u)||_{H^1(\Omega)} \leqslant \frac{\Lambda_L + k_0(|\Omega|^{1/2-1/N_*}\chi + |\Omega|^{1/2-1/r}\Lambda)}{\mathcal{K}(2, \mathcal{B})} = C_6$$

with C_6 only depending on $(N_*, \mathcal{B}, \chi, r, \Lambda, \Lambda_L)$ (recall that k_0 only depends on these data).

Since $u = T_{k_0}(u) + S_{k_0}(u)$, we deduce from this last inequality and (26) that

$$||u||_{H^1(\Omega)} \leqslant C_5 + C_6 = C_7,$$

with C_7 only depending on $(N_*, \mathcal{B}, \chi, r, \Lambda, \Lambda_L)$.

Note that we have just proven the estimate on the solution of (3) given in the theorem: if u is a solution of (3), then it is a solution of (17) with $t = 1$ and we have thus $||u||_{H^1(\Omega)} \leqslant C_7$.

Take now $R = C_7 + 1$. For all $t \in [0, 1]$ and all $u \in H^1_{\Gamma_d}(\Omega)$, solution of $u - t\mathcal{F}(u) = 0$, we have $||u||_{H^1(\Omega)} \neq R$; since $\mathcal{F}$ is a compact operator, the Leray–Schauder Topological Degree allows us to see that $\mathcal{F}$ has a fixed point, that is to say, a solution u of (3).

Step 4: a compact application to (4).

Let $\overline{v} \in H^1_{\Gamma_d}(\Omega)$; we have $\mathbf{v} \cdot \nabla\overline{v} \in L^{2N_*/(N_*+2)}(\Omega) \subset (H^1(\Omega))'$; there exists a unique solution $v = \mathcal{G}(\overline{v})$ to

$$\begin{aligned} &v \in H^1_{\Gamma_d}(\Omega), \\ &\int_\Omega A^{\mathrm{T}}\nabla v \cdot \nabla\varphi + \int_\Omega bv\varphi + \int_{\Gamma_f} \lambda v\varphi\,\mathrm{d}\sigma \\ &\quad = \langle L, \varphi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)} - \int_\Omega \varphi\mathbf{v} \cdot \nabla\overline{v}, \quad \forall\varphi \in H^1_{\Gamma_d}(\Omega). \end{aligned} \tag{27}$$

This defines an application $\mathcal{G} : H^1_{\Gamma_d}(\Omega) \to H^1_{\Gamma_d}(\Omega)$. It is quite easy to see that $\mathcal{G}$ is continuous; indeed, if $\overline{v}_n \to \overline{v}$ in $H^1_{\Gamma_d}(\Omega)$, then $\mathbf{v} \cdot \nabla\overline{v}_n \to \mathbf{v} \cdot \nabla\overline{v}$ in $(H^1(\Omega))'$, which implies $\mathcal{G}(\overline{v}_n) \to \mathcal{G}(\overline{v})$ in $H^1(\Omega)$.

We will now prove that $\mathcal{G}$ is a compact operator. Suppose that $(\overline{v}_n)_{n\geqslant 1}$ is bounded in $H^1_{\Gamma_d}(\Omega)$; then $(\mathbf{v}\cdot\nabla\overline{v}_n)_{n\geqslant 1}$ is bounded in $(H^1(\Omega))'$ so that, using $\varphi = \mathcal{G}(\overline{v}_n) = v_n$ in the equation satisfied by v_n, we get

$$\mathcal{K}(2, \mathcal{B})||v_n||^2_{H^1(\Omega)} \leqslant (\Lambda_L + ||\mathbf{v} \cdot \nabla\overline{v}_n||_{(H^1(\Omega))'})||v_n||_{H^1(\Omega)},$$

which implies that $(v_n)_{n\geqslant 1}$ is bounded in $H^1(\Omega)$.

Up to a subsequence, we can thus suppose that $(v_n)_{n\geqslant 1}$ converges a.e. on Ω and is bounded in $L^{2N_*/(N_*-2)}(\Omega)$. Let $n \geqslant 1$, $m \geqslant 1$; by substracting the equation satisfied by v_m to the equation satisfied by v_n and using $\varphi = v_n - v_m$ as a test function, we get

$$\begin{aligned}
&\mathcal{K}(2, \mathcal{B})||v_n - v_m||^2_{H^1(\Omega)}\\
&\quad\leqslant \left|\int_\Omega (v_n - v_m)\mathbf{v} \cdot (\nabla\overline{v}_m - \nabla\overline{v}_n)\right|\\
&\quad\leqslant 2\sup_{k\geqslant 1} ||\overline{v}_k||_{H^1(\Omega)} \times || \, |v_n\mathbf{v} - v_m\mathbf{v}| \, ||_{L^2(\Omega)}.
\end{aligned}\tag{28}$$

Since $\mathbf{v} \in (L^{N_*}(\Omega))^N$ and $(v_n)_{n\geqslant 1}$ is a bounded sequence of $L^{2N_*(N_*-2)}(\Omega)$ which converges a.e. on Ω, Lemma A.1 tells us that $(v_n\mathbf{v})_{n\geqslant 1}$ converges in $(L^2(\Omega))^N$ and thus is a Cauchy sequence in this space. We deduce from (28) that $(v_n)_{n\geqslant 1}$ is a Cauchy sequence in $H^1_{\Gamma_d}(\Omega)$ and converges in this space.

Since $\mathcal{G}$ is a compact operator, to prove that it has a fixed point, we just have to find $R > 0$ such that, for all $t \in [0, 1]$, there exists no solution of $v - t\mathcal{G}(v) = 0$ satisfying $||v||_{H^1(\Omega)} = R$.

Step 5: estimate on the solutions of $v - t\mathcal{G}(v) = 0$.

Let $t \in [0, 1]$ and suppose that $v \in H^1_{\Gamma_d}(\Omega)$ satifies $v = t\mathcal{G}(v)$. We have then

$$\begin{aligned}
&v \in H^1_{\Gamma_d}(\Omega),\\
&\int_\Omega A^{\mathrm{T}}\nabla v \cdot \nabla\varphi + t\int_\Omega \varphi\mathbf{v} \cdot \nabla v + \int_\Omega bv\varphi + \int_{\Gamma_f} \lambda v\varphi \, \mathrm{d}\sigma\\
&\quad = \langle tL, \varphi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}, \quad \forall\varphi \in H^1_{\Gamma_d}(\Omega).
\end{aligned}\tag{29}$$

Since, for all $t \in [0, 1]$, $t\mathbf{v} \in B(N_*, \chi) + B(r, \Lambda)$, there exists, by the result of Step 3, C_8 only depending on $(N_*, \mathcal{B}, \chi, r, \Lambda)$ such that, for all $\theta \in (H^1_{\Gamma_d}(\Omega))'$ satisfying $||\theta||_{(H^1_{\Gamma_d}(\Omega))'} \leqslant 1$, we can find a solution u to

$$\begin{aligned}
&u \in H^1_{\Gamma_d}(\Omega), ||u||_{H^1(\Omega)} \leqslant C_8,\\
&\int_\Omega A\nabla u \cdot \nabla\varphi + t\int_\Omega u\mathbf{v} \cdot \nabla\varphi + \int_\Omega bu\varphi + \int_{\Gamma_f} \lambda u\varphi \, \mathrm{d}\sigma\\
&\quad = \langle \theta, \varphi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}, \quad \forall\varphi \in H^1_{\Gamma_d}(\Omega).
\end{aligned}\tag{30}$$

By taking $\varphi = v$ in the equation satisfied by u and $\varphi = u$ in the equation satisfied by v, we get

$$\langle \theta, v\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)} = \langle tL, u\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)} \leqslant \Lambda_L C_8.$$

Since this inequality is satisfied for all $\theta \in (H^1_{\Gamma_d}(\Omega))'$ such that $||\theta||_{(H^1_{\Gamma_d}(\Omega))'} \leqslant 1$, we deduce that $||v||_{H^1(\Omega)} \leqslant \Lambda_L C_8$.

Note that this gives the estimate of the theorem; indeed, if v is a solution of (4), then it is a solution of (29) with $t = 1$ so that $||v||_{H^1(\Omega)} \leqslant \Lambda_L C_8$.

Take now $R = \Lambda_L C_8 + 1$. We have just proven that, for any $t \in [0, 1]$, any solution v to $v - t\mathcal{G}(v) = 0$ satisfies $||v||_{H^1(\Omega)} < R$; thus, by the Leray–Schauder Topological Degree, $\mathcal{G}$ has a fixed point, that is to say, a solution of (4).

Step 6: uniqueness.

Since (3) is a linear problem, it suffices to prove that the only solution to (3) with $L = 0$ is the null function. Let u be a solution to (3) with $L = 0$; let v be a solution of (4) with $L = \mathrm{sgn}(u) \in (H^1_{\Gamma_d}(\Omega))'$ (the existence of a solution to this problem is ensured by step 5); by putting $\varphi = v$ in the equation satisfied by u and $\varphi = u$ in the equation satisfied by v, we get $\int_\Omega |u| = 0$, that is, $u = 0$.

A similar reasoning gives the uniqueness of the solution to (4).

2.2. EXISTENCE AND UNIQUENESS IN A NONLINEAR CASE

To prove the existence of a solution to (3), we have not really used the linearity with respect to u of the divergence part $\mathrm{div}(u\mathbf{v})$ (indeed, the tool used in the preceding proof – the Leray–Schauder Topological Degree – is a nonlinear tool). With exactly the same reasoning as in the first three steps of the proof of Theorem 2.1, we can prove the following result.

THEOREM 2.2. *Under hypotheses* (8)–(11), (13), (14), *if* $\Phi\colon \Omega \times \mathbb{R} \to \mathbb{R}^N$ *is a Caratheodory function satisfying*

$$\begin{aligned}&\exists g \in L^{N_*}(\Omega) \text{ such that}\\ &|\Phi(x, s)| \leqslant g(x)(1 + |s|) \quad \text{for a.e. } x \in \Omega, \text{for all } s \in \mathbb{R},\end{aligned} \tag{31}$$

and if Λ_L *is an upper bound of* $||L||_{(H^1_{\Gamma_d}(\Omega))'}$, *there exists a solution to*

$$\begin{aligned}&u \in H^1_{\Gamma_d}(\Omega),\\ &\int_\Omega A\nabla u \cdot \nabla\varphi + \int_\Omega \Phi(\cdot, u) \cdot \nabla\varphi + \int_\Omega bu\varphi + \int_{\Gamma_f} \lambda u\varphi \,\mathrm{d}\sigma\\ &\quad = \langle L, \varphi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}, \quad \forall \varphi \in H^1_{\Gamma_d}(\Omega)\end{aligned} \tag{32}$$

such that $||u||_{H^1(\Omega)} \leqslant C$, *with* C *only depending*[⋆] *on* $(N_*, \mathcal{B}, g, \Lambda_L)$.

⋆ As in Theorem 2.1, C does not depend on g only through $||g||_{L^{N_*}(\Omega)}$, but this dependance could be precised by cutting g into two parts – one small in $L^{N_*}(\Omega)$, the other in $L^r(\Omega)$ for a $r > N$.

Notice, however, that the proof of the existence of a solution to (4) strongly used the linearity of the equation (the *a priori* estimate on the solution to (4) comes from a duality argument); thus, with this reasoning, we cannot state an existence result for a nonlinear problem coming from equation (4) (conversely to what we have done in Theorem 2.2 for equation (3)).

Adding a Lipschitz continuity hypothesis on Φ, it is also quite easy to obtain a uniqueness result for (32).

PROPOSITION 2.1. *Under the hypotheses of Theorem* 2.2*, if* Φ *satisfies*

$$\begin{aligned}&\exists C>0, \exists h\in L^{N_*}(\Omega) \text{ such that}\\ &|\Phi(x,s)-\Phi(x,t)|\leqslant C\big(h(x)+|s|^{2/(N_*-2)}+|t|^{2/(N_*-2)}\big)|s-t|\\ &\quad\text{for a.e. } x\in\Omega, \text{for all } (s,t)\in\mathbb{R}^2,\end{aligned}$$

then the solution to (32) *is unique.*

Proof. Take two solutions u and $\overline{u}$ to (32) and define

$$\mathbf{v}(x)=\begin{cases}\dfrac{\Phi(x,u(x))-\Phi(x,\overline{u}(x))}{u(x)-\overline{u}(x)} & \text{when } u(x)\neq\overline{u}(x),\\ 0 & \text{when } u(x)=\overline{u}(x).\end{cases}$$

Thanks to the Lipschitz continuity hypothesis on Φ, and since $(u,\overline{u})\in H^1(\Omega)\subset L^{2N_*/(N_*-2)}(\Omega)$, we have $\mathbf{v}\in(L^{N_*}(\Omega))^N$; substracting the equation satisfied by $\overline{u}$ to the equation satisfied by u, we see that $w=u-\overline{u}$ satisfies (3) with $L=0$. Since the solution to (3) is unique, this gives $w=0$, that is, $u=\overline{u}$. □

Thanks to the existence, uniqueness and estimates results of Theorem 2.1, we could also prove, as it is classical in the coercive case, the existence results for some other nonlinear equations built from (3) and (4).

3. Regularity Results

In the coercive case, where the right-hand side satisfies*

$$\exists p>N \text{ such that } L\in(W^{1,p'}_{\Gamma_d}(\Omega))', \tag{33}$$

(and under additional properties of $\mathbf{v}$, b, λ, and Γ_d) we already know that the solutions to (3) and (4) are Hölder continuous (see [9] in the pure Dirichlet case, and [6] for other boundary conditions and a convection term in conservative form). We see that this property is still true in the non-coercive case.

* There is a little abuse of notation here. By writing “the right-hand side satisfies (33)”, we mean that we solve (3) or (4) with $L=\widetilde{L}_{|H^1_{\Gamma_d}(\Omega)}$ for a $\widetilde{L}\in(W^{1,p'}_{\Gamma_d}(\Omega))'$; in what follows, we make this abuse of notation by confusing L with $\widetilde{L}$. Under hypothesis (42), this is not an abuse since we can then prove that $H^1_{\Gamma_d}(\Omega)$ is densely imbedded in $W^{1,p'}_{\Gamma_d}(\Omega)$.

3.1. L^∞ BOUND

In the proof of Theorem 2.1, the role played by the convection term in conservative form $\operatorname{div}(\mathbf{v}u)$ is quite different than the role played by the convection term in non-conservative form $\mathbf{v}\cdot\nabla u$ (the technique used to obtain estimates of the solution to (3) does not work to obtain estimates of the solution to (4)).

As shown in [9], when considering regularity results, the difference between (3) and (4) is even more stronger; when the convection term is in the non-conservative form, hypothesis (12) is enough, but when it is in the conservative form, $\mathbf{v}$ must be (at least for technical reasons) slightly more integrable than what is strictly necessary to obtain the existence result.

That is why we will have to consider, when dealing with (3), the following hypothesis:*

$$\exists r > N \text{ such that } \mathbf{v} \in (L^r(\Omega))^N. \tag{34}$$

When $\mathbf{v}$ satisfies this hypothesis, we denote by $\Lambda_{\mathbf{v}}$ an upper bound of $||\,|\mathbf{v}|\,||_{L^r(\Omega)}$.

The first regularity results deal with essential bounds on the solutions to (3) and (4)

PROPOSITION 3.1. *Under hypotheses* (8)–(11), (14), (33), *and* (34), *the solution u to* (3) *is in $L^\infty(\Omega)$. Moreover, if Λ_L is an upper bound of $||L||_{(W^{1,p'}_{\Gamma_d}(\Omega))'}$, there exists C only depending on $(N_*, \mathcal{B}, r, \Lambda_{\mathbf{v}}, p, \Lambda_L)$ such that $||u||_{L^\infty(\Omega)} \leqslant C$.*

PROPOSITION 3.2. *Under hypotheses* (8)–(12), (14), *and* (33), *the solution v to* (4) *is in $L^\infty(\Omega)$. Moreover, if $r > N$ and $\Lambda \geqslant 0$ are such that $\mathbf{v} \in B(N_*, \chi) + B(r, \Lambda)$, with χ satisfying* (15), *and if Λ_L is an upper bound of $||L||_{(W^{1,p'}_{\Gamma_d}(\Omega))'}$, then there exists C only depending on $(N_*, \mathcal{B}, \chi, r, \Lambda, p, \Lambda_L)$ such that $||v||_{L^\infty(\Omega)} \leqslant C$.*

Proof of Proposition 3.1. The solution u of (3) is also a solution of (17) with $t = 1$. Since $\mathbf{v} \in B(N_*, 0) + B(r, \Lambda_{\mathbf{v}})$, the reasoning in the proof of Theorem 2.1 that has lead to (25) can be applied to u with $\chi = 0$; thus, there exists $k_0 > 0$ only depending on $(N_*, \mathcal{B}, r, \Lambda_{\mathbf{v}}, p, \Lambda_L)$ (note that $|\Omega|^{1/2-1/p}\Lambda_L$ is an upper bound of $||L||_{(H^1_{\Gamma_d}(\Omega))'}$) such that, for all $k \geqslant k_0$,

$$||S_k(u)||_{H^1(\Omega)} \leqslant \frac{2}{\mathcal{K}(2,\mathcal{B})}\left(\frac{|\langle L, S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}|}{||S_k(u)||_{H^1(\Omega)}} + \Lambda_{\mathbf{v}} k |E_k|^{1/2-1/r}\right), \tag{35}$$

with $S_k(u) = u - T_k(u) = u - \min(k, \max(u, -k))$ and $E_k = \{x \in \Omega \mid |u(x)| \geqslant k\}$.

* One can notice that, in dimension $N = 2$, (12) implies (34) (i.e., there is no additional hypothesis on $\mathbf{v}$ with respect to the hypotheses of Theorem 2.1).

Since $S_k(u) = 0$ outside E_k and $p' < 2$, we have $||S_k(u)||_{W^{1,p'}(\Omega)} \leqslant |E_k|^{1/p'-1/2} \times ||S_k(u)||_{H^1(\Omega)}$, so that

$$\frac{|\langle L, S_k(u)\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}|}{||S_k(u)||_{H^1(\Omega)}} \leqslant \Lambda_L \frac{||S_k(u)||_{W^{1,p'}(\Omega)}}{||S_k(u)||_{H^1(\Omega)}} \leqslant \Lambda_L |E_k|^{1/2-1/p}. \tag{36}$$

Let $h > k \geqslant k_0$. Since $|S_k(u)| \geqslant (h-k)$ in E_h, and thanks to the Sobolev injection $W^{1,1}(\Omega) \hookrightarrow L^{N/(N-1)}(\Omega)$, there exists C_1 only depending on Ω such that

$$\begin{aligned}(h-k)|E_h|^{(N-1)/N} &\leqslant ||S_k(u)||_{L^{N/(N-1)}(\Omega)} \\ &\leqslant C_1 ||S_k(u)||_{W^{1,1}(\Omega)} \\ &\leqslant C_1 |E_k|^{1/2} ||S_k(u)||_{H^1(\Omega)}.\end{aligned} \tag{37}$$

(36) and (37) used in (35) give then, for all $h > k \geqslant k_0$,

$$\begin{aligned}|E_h|^{(N-1)/N} &\leqslant \frac{2C_1|E_k|^{1/2}}{\mathcal{K}(2,\mathcal{B})(h-k)}\big(\Lambda_L|E_k|^{1/2-1/p} + \Lambda_{\mathbf{v}} k |E_k|^{1/2-1/r}\big) \\ &\leqslant \frac{C_2}{h-k}\big(|E_k|^{1-1/p} + k|E_k|^{1-1/r}\big),\end{aligned}$$

with C_2 only depending on $(\mathcal{B}, \Lambda_{\mathbf{v}}, \Lambda_L)$. Since, for $q \in \{r, p\}$, $|E_k|^{1-1/q} \leqslant |\Omega|^{1/(\inf(r,p))-1/q}|E_k|^{1-1/(\inf(r,p))}$, there exists C_3 only depending on $\mathcal{B}$, r, $\Lambda_{\mathbf{v}}$, p, and Λ_L such that, for all $h > k \geqslant k_0$,

$$|E_h| \leqslant \frac{C_3^\beta (1+k)^\beta}{(h-k)^\beta}|E_k|^\gamma$$

with $\beta = \frac{N}{N-1} > 0$ and $\gamma = \beta(1 - \frac{1}{\inf(r,p)}) > 1$ (recall that $r > N$ and $p > N$).

For all $h > k \geqslant 0$, we have then

$$|E_{h+k_0}| \leqslant \frac{C_3^\beta(1+k_0)^\beta(1+k)^\beta}{(h-k)^\beta}|E_{k+k_0}|^\gamma$$

(because $(1+k+k_0) \leqslant (1+k_0)(1+k)$), and Lemma A.2 (a generalization of the classical lemma of Stampacchia) applied to $F(k) = |E_{k+k_0}|$ gives H only depending on $(C_3, k_0, \beta, \gamma, \Omega)$ (note that $F(0) = |E_{k_0}| \leqslant |\Omega|$), i.e., on $(N_*, \mathcal{B}, r, \Lambda_{\mathbf{v}}, p, \Lambda_L)$, such that $|E_{H+k_0}| = 0$, that is to say, $|u| \leqslant H + k_0$ a.e. on Ω.

Proof of Proposition 3.2. The idea is identical to that of the preceding proof. We write $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$ with $\mathbf{v}_0 \in B(N_*, \chi)$ and $\mathbf{v}_1 \in B(r, \Lambda)$.

Since $vS_k(v) \geqslant (S_k(v))^2$ and $\nabla v = \nabla(S_k(v))$ a.e. on the set $\{S_k(v) \neq 0\}$, using $S_k(v)$ as a test function in (4), we get

$$\begin{aligned}&(\mathcal{K}(2,\mathcal{B}) - \chi C_S(\Omega, N_*))||S_k(v)||^2_{H^1(\Omega)} \\ &\quad\leqslant \int_\Omega A^{\mathrm{T}}\nabla v \cdot \nabla(S_k(v)) + \int_\Omega S_k(v)\mathbf{v}_0 \cdot \nabla v + \int_\Omega bvS_k(v) +\end{aligned}$$

$$+\int_{\Gamma_f} \lambda v S_k(v)\,\mathrm{d}\sigma$$
$$\leqslant |\langle L, S_k(v)\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}| + \Lambda ||S_k(v)||_{L^{2r/(r-2)}(\Omega)} ||S_k(v)||_{H^1(\Omega)}. \quad (38)$$

Since $\frac{2r}{r-2} < \frac{2N}{N-2}$, there exists $q_r > \frac{2r}{r-2}$ only depending on r and N such that $H^1(\Omega) \hookrightarrow L^{q_r}(\Omega)$; denoting by C_1 the norm of this injection (which only depends on r and Ω) and $E_k = \{x \in \Omega \mid |v(x)| \geqslant k\}$, we have then

$$||S_k(v)||_{L^{2r/(r-2)}(\Omega)} \leqslant C_1 |E_k|^{(r-2)/2r - 1/q_r} ||S_k(v)||_{H^1(\Omega)}.$$

But, since $|\Omega|^{1/2-1/p}\Lambda_L$ is an upper bound of $||L||_{(H^1_{\Gamma_d}(\Omega))'}$, there exists, by Theorem 2.1, C_2 only depending on $(N_*, \mathcal{B}, \chi, r, \Lambda, p, \Lambda_L)$ such that $||v||_{H^1(\Omega)} \leqslant C_2$, which implies $|E_k| \leqslant C_2^2/k^2$ for all $k \geqslant 0$. We can thus find $k_0 > 0$ only depending on $(N_*, \mathcal{B}, \chi, r, \Lambda, p, \Lambda_L)$ such that, for all $k \geqslant k_0$, $C_1 \Lambda |E_k|^{(r-2)/2r-1/q_r} \leqslant (\mathcal{K}(2, \mathcal{B}) - \chi C_S(\Omega, N_*))/2$.

We have then, thanks to (38) and when $k \geqslant k_0$,

$$||S_k(v)||_{H^1(\Omega)} \leqslant \frac{2}{\mathcal{K}(2, \mathcal{B}) - \chi C_S(\Omega, N_*)} \times \frac{|\langle L, S_k(v)\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)}|}{||S_k(v)||_{H^1(\Omega)}}.$$

This inequality is similar to (35) (it is even simpler), and we can then conclude as in the proof of Proposition 3.1.

3.2. HÖLDER CONTINUITY

To state the Hölder continuity results, we need (at least for technical reasons) stronger integrability hypotheses on b and λ; thus we replace (10) and (11) by

$$\begin{aligned} &\exists \overline{r} > N \text{ such that } b \in L^{\overline{r}/2}(\Omega),\ \lambda \in L^{\overline{r}-1}(\partial\Omega) \\ &\text{and } b \geqslant 0 \text{ a.e. on } \Omega,\ \lambda \geqslant 0\ \sigma\text{-a.e. on } \partial\Omega. \end{aligned} \quad (39)$$

We denote by Λ_b an upper bound of $||b||_{L^{\overline{r}/2}(\Omega)}$, and by Λ_λ an upper bound of $||\lambda||_{L^{\overline{r}-1}(\partial\Omega)}$.

We also need a hypothesis on Γ_d and Γ_f; these sets must be "well-distributed" on $\partial\Omega$. Thus we introduce two kinds of mappings of $\partial\Omega$:

$$\begin{aligned} &O \text{ is an open subset of } \mathbb{R}^N, \\ &h\colon O \to B := \{x \in \mathbb{R}^N \mid |x| < 1\} \text{ is a Lipschitz continuous} \\ &\text{homeomorphism with a Lipschitz continuous inverse mapping,} \\ &h(O \cap \Omega) = B_+ := \{x \in B \mid x_N > 0\}, \\ &h(O \cap \partial\Omega) = \{x \in \partial B_+ \mid x_N = 0\}, \end{aligned} \quad (40)$$

$$\begin{array}{l} O \text{ is an open subset of } \mathbb{R}^N, \\ h\colon O \to B \text{ is a Lipschitz continuous homeomorphism} \\ \text{ with a Lipschitz continuous inverse mapping,} \\ h(O \cap \Omega) = B_{++} := \{x \in B \mid x_N > 0\,,\ x_{N-1} > 0\}, \\ h(O \cap \Gamma_f) = \{x \in \partial B_{++} \mid x_{N-1} = 0\}, \\ h(O \cap \Gamma_d) = \{x \in \partial B_{++} \mid x_N = 0\}, \end{array} \tag{41}$$

and we suppose that there exists a finite number of $(O_i, h_i)_{i\in[1,m]}$ such that

$$\begin{array}{l} \partial\Omega \subset \bigcup_{i=1}^m O_i \text{ and, for all } i \in [1, m], \\ (O_i, h_i) \text{ is of one of the following types:} \\ (D) \quad O_i \cap \partial\Omega = O_i \cap \Gamma_d \text{ and } (O_i, h_i) \text{ satisfies (40)} \\ (F) \quad O_i \cap \partial\Omega = O_i \cap \Gamma_f \text{ and } (O_i, h_i) \text{ satisfies (40)} \\ (DF) \ \ (O_i, h_i) \text{ satisfies (41).} \end{array} \tag{42}$$

COROLLARY 3.1. *Under hypotheses* (8), (9), (14), (33), (34), (39), *and* (42), *the solution u to* (3) *is Hölder continuous on* Ω. *More precisely, if Λ_L is an upper bound of* $||L||_{(W^{1,p'}_{\Gamma_d}(\Omega))'}$, *then there exists $\kappa > 0$ only depending on* $(\Omega, \alpha_A, \Lambda_A, \overline{r}, r, p)$ *and C only depending on*

$$(N_*, \mathcal{B}, \Lambda_A, \overline{r}, \Lambda_b, \Lambda_\lambda, r, \Lambda_{\mathbf{v}}, p, \Lambda_L)$$

such that u satisfies[⋆] $||u||_{\mathcal{C}^{0,\kappa}(\Omega)} \leqslant C$.

Note that, provided that the function g in (31) is in $L^r(\Omega)$ for a $r > N$, the results of Proposition 3.1 and Corollary 3.1 are also true for any solution of (32).

COROLLARY 3.2. *Under hypotheses* (8), (9), (12), (14), (33), (39), *and* (42), *the solution v to* (4) *is Hölder continuous on* Ω. *More precisely, if Λ_L is an upper bound of* $||L||_{(W^{1,p'}_{\Gamma_d}(\Omega))'}$, $r > N$ *and* $\Lambda \geqslant 0$, *then there exists* $\eta > 0$ *only depending on* (N_*, Ω, α_A), $\kappa > 0$ *only depending on* $(N_*, \Omega, \alpha_A, \Lambda_A, \overline{r}, r, \Lambda, p)$, *and C only depending on*

$$(N_*, \mathcal{B}, \Lambda_A, \overline{r}, \Lambda_b, \Lambda_\lambda, r, \Lambda, p, \Lambda_L)$$

such that, when $\mathbf{v} \in B(N_*, \eta) + B(r, \Lambda)$, *$v$ satisfies* $||v||_{\mathcal{C}^{0,\kappa}(\Omega)} \leqslant C$.

Proof of Corollaries 3.1 and 3.2. Due to Proposition 3.1 (respectively, 3.2), the solution u to (3) (respectively, v to (4)) is essentially bounded on Ω, and we have an estimate of its L^∞ norm. Thus, due to (34) and (39) (respectively, (39)), the terms $\varphi \to \int_\Omega u\varphi$, $\varphi \to \int u\mathbf{v}\cdot\nabla\varphi$, $\varphi \to \int_\Omega bu\varphi$ and $\varphi \to \int_{\Gamma_f} \lambda u\varphi \,\mathrm{d}\sigma$ (respectively, $\varphi \to \int_\Omega bv\varphi$ and $\varphi \to \int_{\Gamma_f} \lambda v\varphi\,\mathrm{d}\sigma$) are in $(W^{1,\inf(r,\overline{r})'}_{\Gamma_d}(\Omega))'$ (respectively, $(W^{1,\overline{r}'}_{\Gamma_d}(\Omega))'$), and we have a bound on their norms in this space.

⋆ We denote by $\mathcal{C}^{0,\kappa}(\Omega)$ the space of κ-Hölder continuous functions, endowed with its usual norm.

By putting these terms on the right-hand side, we notice then that u satisfies

$$\begin{aligned}&u \in H^1_{\Gamma_d}(\Omega),\\&\int_\Omega A\nabla u \cdot \nabla\varphi + \int_\Omega u\varphi = \langle \widetilde{L}, \varphi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)},\\&\quad\forall \varphi \in H^1_{\Gamma_d}(\Omega),\end{aligned} \tag{43}$$

with $\widetilde{L} \in (W^{1,l'}_{\Gamma_d}(\Omega))'$ for $l = \inf(\overline{r}, r, p) > N$. The results of [9] (in the pure Dirichlet case) or of [6] (for other boundary conditions) give the Hölder continuity of u, as well as the estimates of the Hölder space to which u belongs and of its norm in this space.

For v, we get an equation of the kind

$$\begin{aligned}&v \in H^1_{\Gamma_d}(\Omega),\\&\int_\Omega A^{\mathrm{T}}\nabla v \cdot \nabla\varphi + \int_\Omega \varphi\mathbf{v} \cdot \nabla v = \langle \widetilde{L}, \varphi\rangle_{(H^1_{\Gamma_d}(\Omega))', H^1_{\Gamma_d}(\Omega)},\\&\quad\forall \varphi \in H^1_{\Gamma_d}(\Omega),\end{aligned} \tag{44}$$

with $\widetilde{L} \in (W^{1,l'}_{\Gamma_d}(\Omega))'$ for $l = \inf(\overline{r}, p) > N$. In the pure Dirichlet case, the results of [9] give then the Hölder continuity of v; for other boundary conditions, a slight modification of the methods in [6] gives the Hölder continuity (as well as the estimates) of v.

4. The Duality Method for Non-Regular Right-Hand Sides

As it is shown in [9], the regularity results of Corollaries 3.1 and 3.2 can be transformed into existence and uniqueness results for weaker right-hand sides.

We suppose here hypotheses (8), (9), (12), (14), (39), and (42).

Define $\mathcal{T}: (H^1_{\Gamma_d}(\Omega))' \to H^1_{\Gamma_d}(\Omega)$ such that, for all $L \in (H^1_{\Gamma_d}(\Omega))'$, $\mathcal{T}L$ is the unique solution to (4). According to Theorem 2.1, $\mathcal{T}$ is well defined, linear and continuous.

Let $p \in]N, \infty[$. Due to Corollary 3.2,

$$\mathcal{T}_p = \mathcal{T}_{|(W^{1,p'}_{\Gamma_d}(\Omega))'}: \left(W^{1,p'}_{\Gamma_d}(\Omega)\right)' \to H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$$

is well defined, linear and continuous.* The adjoint operator of $\mathcal{T}_p$ is a linear continuous application $\mathcal{T}_p^*: (H^1(\Omega) \cap \mathcal{C}(\overline{\Omega}))' \to W^{1,p'}_{\Gamma_d}(\Omega)$ (since $1 < p < \infty$, $W^{1,p'}_{\Gamma_d}(\Omega)$ is a reflexive space).

Let $\mathcal{M}(\overline{\Omega}) = (\mathcal{C}(\overline{\Omega}))'$ (identified through the Riesz representation theorem to the space of bounded measures on $\overline{\Omega}$). Since $H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is continuously and densely embedded in $\mathcal{C}(\overline{\Omega})$ and in $H^1(\Omega)$, $\mathcal{M}(\overline{\Omega})$ and $(H^1(\Omega))'$ are continuously embedded in $(H^1(\Omega) \cap \mathcal{C}(\overline{\Omega}))'$.

* $H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is endowed with the norm $\|\cdot\|_{H^1(\Omega)} + \|\cdot\|_{\mathcal{C}(\overline{\Omega})}$.

Thus, we can talk of $\mathcal{M}(\overline{\Omega}) + (H^1(\Omega))'$ as a subspace* of $(H^1(\Omega) \cap \mathcal{C}(\overline{\Omega}))'$.
Let $\zeta \in \mathcal{M}(\overline{\Omega}) + (H^1(\Omega))'$. By definition, $f_p = \mathcal{T}_p^* \zeta$ is the unique solution to

$$\begin{aligned} &f_p \in W^{1,p'}_{\Gamma_d}(\Omega), \quad \forall L \in (W^{1,p'}_{\Gamma_d}(\Omega))', \\ &\langle L, f_p \rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))', W^{1,p'}_{\Gamma_d}(\Omega)} \\ &\quad = \langle \zeta, \mathcal{T}_p L \rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))', H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})} \\ &\quad = \langle \zeta, \mathcal{T} L \rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))', H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}. \end{aligned} \tag{45}$$

Take now $q \in]N, p[$ and f_q as the solution of (45) when p is replaced by q. Let $L \in (W^{1,p'}_{\Gamma_d}(\Omega))'$; since $W^{1,q'}_{\Gamma_d}(\Omega) \hookrightarrow W^{1,p'}_{\Gamma_d}(\Omega)$, $f_q \in W^{1,p'}_{\Gamma_d}(\Omega)$ and $L_{|W^{1,q'}_{\Gamma_d}(\Omega)} \in (W^{1,q'}_{\Gamma_d}(\Omega))'$, so that, by definition of f_q,

$$\begin{aligned} \langle L, f_q \rangle_{(W^{1,p'}_{\Gamma_d}(\Omega))', W^{1,p'}_{\Gamma_d}(\Omega)} &= \langle L_{|W^{1,q'}_{\Gamma_d}(\Omega)}, f_q \rangle_{(W^{1,q'}_{\Gamma_d}(\Omega))', W^{1,q'}_{\Gamma_d}(\Omega)} \\ &= \langle \zeta, \mathcal{T} L \rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))', H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}. \end{aligned}$$

Thus, f_q is also a solution to (45) and we have then $f_q = f_p$ for all $q \in]N, p[$.

Thus, the solution to (45) belongs to $\bigcap_{q<N/(N-1)} W^{1,q}_{\Gamma_d}(\Omega)$ and is in fact the unique solution to

$$\begin{aligned} &f \in \bigcap_{q<N/(N-1)} W^{1,q}_{\Gamma_d}(\Omega), \quad \forall q < \frac{N}{N-1}, \forall L \in \left(W^{1,q}_{\Gamma_d}(\Omega)\right)', \\ &\langle L, f \rangle_{(W^{1,q}_{\Gamma_d}(\Omega))', W^{1,q}_{\Gamma_d}(\Omega)} = \langle \zeta, \mathcal{T} L \rangle_{(H^1(\Omega)\cap\mathcal{C}(\overline{\Omega}))', H^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}. \end{aligned} \tag{46}$$

The unique solution to (46) is called the duality solution of

$$\begin{aligned} &-\mathrm{div}(A\nabla f) - \mathrm{div}(\mathbf{v} f) + bf = \zeta && \text{in } \Omega, \\ &f = 0 && \text{on } \Gamma_d, \\ &A\nabla f \cdot \mathbf{n} + (\lambda + \mathbf{v}\cdot\mathbf{n}) f = 0 && \text{on } \Gamma_f. \end{aligned} \tag{47}$$

This gives a notion of the solution to (1) when the right-hand side L is in $\mathcal{M}(\overline{\Omega}) + (H^1(\Omega))'$, for which we have the existence and uniqueness (as well as estimates, since $\mathcal{T}_p^*$ is linear continuous – its norm is that of $\mathcal{T}_p$, which can be bounded using the results of Theorem 2.1 and Corollary 3.2).

To understand why, by solving (46), we can say that, in a way, we have solved (47), we refer the reader to [6]. In particular, it is quite easy to see that, when $\zeta \in (H^1(\Omega))'$, the solution to (3) with $L = \zeta_{|H^1_{\Gamma_d}(\Omega)}$ is the solution to (46); we can also state integral formulations (one equivalent to (46), the other weaker than (46)) satisfied by the solution of (46) that makes it easier to see why this solution is the solution to (47).

---

* Endowed with the norm $||\zeta|| = \inf\{||\mu||_{\mathcal{M}(\overline{\Omega})} + ||L||_{(H^1(\Omega))'}, (\mu, L) \in \mathcal{M}(\overline{\Omega}) \times (H^1(\Omega))', \mu + L = \zeta\}$, this is a Banach space, and it is continuously embedded in $(H^1(\Omega) \cap \mathcal{C}(\overline{\Omega}))'$. In fact, one can show that $(H^1(\Omega) \cap \mathcal{C}(\overline{\Omega}))' = \mathcal{M}(\overline{\Omega}) + (H^1(\Omega))'$.

Under hypothesis (34), one can do the same reasoning, using the regularity results on the solution to (3). In this case, we obtain a duality solution to

$$\begin{cases} -\mathrm{div}(A^{\mathrm{T}}\nabla f) + \mathbf{v}\cdot\nabla f + bf = \zeta & \text{in } \Omega, \\ f = 0 & \text{on } \Gamma_d, \\ A^{\mathrm{T}}\nabla f\cdot\mathbf{n} + \lambda f = 0 & \text{on } \Gamma_f. \end{cases} \tag{48}$$

All the results on the duality solutions obtained in [6] also do apply here; in particular, we could state a stability result similar to the one of Theorem 4.1 in [6].

Acknowledgement

The author wishes to thank Lucio Boccardo for his invaluable help.

Appendix A. Technical Lemmas

LEMMA A.1. *Let $(p,q,r)\in[1,\infty]$ such that $q<\infty$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. If $g\in L^q(\Omega)$ and $(f_n)_{n\geqslant 1}$ is a bounded sequence of $L^p(\Omega)$ which converges a.e. on Ω to f, then $f_ng\to fg$ in $L^r(\Omega)$.*

REMARK A.1. This result is also true when Ω is replaced by any measured space $(X,\mathcal{A},\mu)$.

Proof of Lemma A.1. We have $f_ng\to fg$ a.e. on Ω. Since $r<\infty$ (because $q<\infty$) and Ω is of finite measure, thanks to the Vitali Theorem, we just have to prove the r-equi-integrability of $(f_ng)_{n\geqslant 1}$ to get the convergence in $L^r(\Omega)$ of this sequence.

Denote by M an upper bound of $(||f_n||_{L^p(\Omega)})_{n\geqslant 1}$. Let E be a measurable subset of Ω; by the Hölder inequality, we have

$$||f_ng||_{L^r(E)}\leqslant ||f_n||_{L^p(E)}||g||_{L^q(E)}\leqslant M||g||_{L^q(E)}.$$

Since $q<\infty$, we have $||g||_{L^q(E)}\to 0$ as $|E|\to 0$; this gives the r-equi-integrability of $(f_ng)_{n\geqslant 1}$ and concludes the proof of this lemma. □

LEMMA A.2. *Let $F\colon\mathbb{R}^+\to\mathbb{R}^+$ be a non-increasing function. If there exist $\beta>0$, $\gamma>1$ and $C>0$ such that*

$$\forall h>k\geqslant 0,\quad F(h)\leqslant\frac{C^\beta(1+k)^\beta}{(h-k)^\beta}F(k)^\gamma$$

and if

$$H=\exp\left(\sum_{n\geqslant 0}\frac{2^{1/\beta}CF(0)^{(\gamma-1)/\beta}}{(2^{(\gamma-1)/\beta})^n}\right)<+\infty,$$

then $F(H)=0$.

For the proof of this variant of Lemma A.1(i) in [9], we refer the reader to [6].

References

1. Adams, R. A.: *Sobolev Spaces*, Accademic Press, 1975.
2. Guibe, O. and Ben Cheikh, M.: 'Résultats d'existence et d'unicité pour une classe de problèmes non linéaires et non coercitifs', *C. R. Acad. Sci. Paris Sér. I Math.* **329**(11) (1999), 967–972.
3. Boccardo, L.: 'Some nonlinear Dirichlet problems in L^1 involving lower order term in divergence form', in *Progress in Elliptic and Parabolic Partial Differential Equations*, Capri, 1994, Pitman Res. Notes Math. Ser. 350, Longman, Harlow, 1996, pp. 43–57.
4. Boccardo, L. and Gallouët, T.: 'Nonlinear elliptic and parabolic equations involving measure data', *J. Funct. Anal.* **87** (1989), 241–273.
5. Deimling, K.: *Nonlinear Functional Analysis*, Springer, 1985.
6. Droniou, J.: 'Solving convection–diffusion equations with mixed, Neumann and Fourier boundary conditions and measures as data, by a duality method', *Adv. Differential Equations* **5**(10–12) (2000), 1341–1396.
7. Droniou, J. and Gallouët, T.: 'A uniqueness result for quasilinear elliptic equations with measures as data', accepted for publication in *Rendiconti di Matematica*.
8. Leray, J. and Lions, J. L.: 'Quelques résultats de Višik sur les problèmes elliptiques semilinéaires par les méthodes de Minty et Browder', *Bull. Soc. Math. France* **93** (1965), 97–107.
9. Stampacchia, G.: 'Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus', *Ann. Inst. Fourier (Grenoble)* **15** (1965), 189–258.