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Is the Regge Calculus a consistent approximation to General Relativity?

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Abstract

We will ask the question of whether or not the Regge calculus (and two related simplicial formulations) is a consistent approximation to General Relativity. Our criteria will be based on the behaviour of residual errors in the discrete equations when evaluated on solutions of the Einstein equations. We will show that for generic simplicial lattices the residual errors can not be used to distinguish metrics which are solutions of Einstein's equations from those that are not. We will conclude that either the Regge calculus is an inconsistent approximation to General Relativity or that it is incorrect to use residual errors in the discrete equations as a criteria to judge the discrete equations.

1. Introduction

Since its inception in 1961 the Regge calculus [1] has been believed to be a consistent and convergent approximation to Einstein's theory of General Relativity. It has often been touted as a natural discretisation of Einstein's equations and it has also been used as a possible basis for a quantum theory of gravity (for a detailed bibliography see Williams and Tuckey [2]). However the use of the Regge calculus in numerical relativity has been limited to highly symmetric spacetimes and upon lattices specifically designed for those spacetimes. Yet little is known about how the Regge calculus performs for generic spacetimes and it is this class of spacetimes for which the Regge calculus is most suited.

It is therefore very important that the Regge calculus be tested for non-symmetric spacetimes and upon generic simplicial lattices. Ideally this would entail solving the Regge equations for a variety of non-trivial spacetimes and to compare the solutions against those of the Einstein equations. Unfortunately the task of solving the Regge equations for such spacetimes is way beyond current technology. Thus we are forced to look for some other (less stringent) criteria.

A much simpler test is to evaluate the residual error in the Regge calculus by evaluating the Regge equations for a set of leg-lengths computed from a known solution of Einstein's equations.

For a typical Regge equation the residual error τ_i can be defined as

$$\tau_i = \sum_{j(i)} \theta_j \frac{\partial A_j}{\partial L_i^2} \tag{1.1}$$

For an exact solution of the Regge equations each τ_i would be zero. However, with the leg lengths set from a solution of Einstein's equations one can not expect the τ_i to be zero. We would expect, though, that as the simplicial lattice is refined then the τ_i should vanish. As we shall see later, what is most important is not that the τ_i vanish but how quickly they vanish with respect to successive refinements in the simplicial lattice.

We will compute the residual errors for three different simplicial lattices and for five different smooth metrics (four of which are exact solutions of Einstein's equations).

The principle result of this paper is that, **the residual error in the Regge equations cannot be used to distinguish between metrics which are solutions of Einstein's equations from those that are not**.

This result has been previously noted by Miller [4], however, we shall draw sharply different conclusions from those made by Miller.

We shall conclude that either the Regge calculus is not a consistent approximation to General Relativity or that it is incorrect to use residual errors as the criteria on which to judge the Regge calculus.

The use of residual errors as a criteria on which to judge a discrete set of equations is commonly used in the analysis of finite difference approximations to differential equations. Its value is that it is easy to apply and that quite often one can show, by analytic methods, that if the residual error vanishes as the step length is reduced and if the finite difference scheme is stable then the exact solution of the discrete equations will converge to solutions of the original differential equations (see for example the Lax Equivalence Theorem [5]).

2. Methods

We will consider in sequence three main issues. First, the algorithm for assigning the leg lengths in the lattice. Second, the choice of simplicial lattices and, finally, the choice of continuum metrics.

Our basic plan is to construct a local group of simplices representing a small patch on the continuum spacetime and to compute the residual errors in the discrete equations as a function of the scale of the patch. However, to evaluate the residual errors one needs to establish some correspondence between the smooth $g_{\mu\nu}$'s of the continuum and the L_{ij} of the lattice. A natural way to do this is to map geodesic segments from the continuum to legs of the lattice. The geodesic length can be computed from the continuum metric and then assigned to the L_{ij}. Thus we are lead to the following two point boundary problem.

Find the curve $x^\mu(\lambda)$ which, for $0 \leq \lambda \leq 1$, satisfies

$$0 = \frac{d^2 x^\mu}{d\lambda^2} + \sum_{\alpha\beta} \Gamma^\mu_{\alpha\beta}(x^\rho(\lambda)) \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

subject to $x^\mu(0) = x^\mu_i$ and $x^\mu(1) = x^\mu_j$.

Once the geodesic is known the squared leg length is assigned as

$$L^2_{ij} = \omega \left(\int_0^1 \left| g_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \right|^{1/2} d\lambda \right)^2$$

with $\omega = \pm 1$ according to the signature of the geodesic (which is known to be constant along the geodesic).

This boundary value problem was solved by a shooting method. The idea is to convert the boundary value problem into an initial value problem and then to try to find the initial values so as to satisfy the end boundary condition. Let $y^\mu(\lambda, g^\nu)$ denote a solution of the initial value problem starting with the guess $g^\mu = dy^\mu/d\lambda$. The strategy now is to solve the coupled equations $0 = x^\mu(1) - y^\mu(1, g^\nu)$ for g^ν. This was done via a Newton-Raphson approach

$$g^\mu_{n+1} = g^\mu_n + \delta g^\mu$$
$$x^\mu(1) - y^\mu(1, g^\nu_n) = -\left(\frac{\partial y^\mu}{\partial g^\alpha}\right)_n \delta g^\alpha$$

The partial derivatives were evaluated numerically

$$\left(\frac{\partial y^\mu}{\partial g^\alpha}\right)_n = \frac{y^\mu(1, g^\nu_n + \delta h^\nu) - y^\mu(1, g^\nu_n - \delta h^\nu)}{2\delta h^\alpha} \tag{2.1}$$

Notice that to evaluate all of the partial derivatives requires at least 32 complete integrations of the initial value problem. The leg lengths were evaluated by appending the differential equation for $ds/d\lambda$ to the Runge-Kutta routine.

We will need to compute the residual errors for series of patches of varying sizes. This is actually rather easy to do. Let a set of coordinates in the patch on the continuum spacetime be given by y^μ and their values at the vertices by y^μ_i. Now define a coordinate transformation from y^μ to x^μ by

$$y^\mu = x^\mu_\star + \epsilon(x^\mu - x^\mu_\star) \qquad 0 < \epsilon \leq 1$$

This will be viewed as an active transformation having the effect of focusing the vertices upon the (freely chosen) central point $x^\mu_\star$. In this construction each vertex carries with it its initial coordinates. The metric in these coordinates is therefore

$$ds^2 = \epsilon^2 g_{\mu\nu}(x^\mu_\star + \epsilon(x^\mu - x^\mu_\star)) dx^\mu dx^\nu$$

The leading ϵ^2 serves only as a constant conformal factor and can thus be absorbed by a re-scaling $ds \leftarrow ds/\epsilon$. Finally, the form of the metric from which the geodesics are calculated is just

$$ds^2 = g_{\mu\nu}(x^\mu_\star + \epsilon(x^\mu - x^\mu_\star))dx^\mu dx^\nu \qquad (2.2)$$

with the coordinates at the vertices x^μ_i being chosen independently of ϵ.

Let us now turn to the choice of simplicial lattices. We will consider just three distinct models. Two of the models were constructed from a set of tetrahedra surrounding a common vertex with the 4-simplices being generated by dragging the common vertex forwards and backwards in time. The third model was a 2x2x2x2 four dimensional hypercubic lattice. The details of these models are as follows.

2.1. Model 1

This model is based on the collection of four tetrahedra described by

$$(1,2,3,5) \quad (1,2,4,5) \quad (1,3,4,5) \quad (2,3,4,5)$$

as depicted in Figure (1). Spatial coordinates are then assigned to each of the five vertices

$$(i) \leftarrow (0, x^2_i, x^3_i, x^4_i).$$

Next, a pair of new vertices $(5^\uparrow)$ and $(5^\downarrow)$ are created with coordinates

$$(5^\uparrow) \leftarrow (+x^1_5, x^2_5, x^3_5, x^4_5)$$
$$(5^\downarrow) \leftarrow (-x^1_5, x^2_5, x^3_5, x^4_5)$$

Finally the set of 4-simplices is created by connecting $(5^\uparrow)$ and $(5^\downarrow)$ to all of the vertices $(1), (2), (3), (4)$ and (5). This creates a simplicial lattice with eight 4-simplices

$$(1,2,3,5,5^\uparrow) \quad (1,2,4,5,5^\uparrow) \quad (1,3,4,5,5^\uparrow) \quad (2,3,4,5,5^\uparrow)$$
$$(1,2,3,5,5^\downarrow) \quad (1,2,4,5,5^\downarrow) \quad (1,3,4,5,5^\downarrow) \quad (2,3,4,5,5^\downarrow)$$

2.2. Model 2

This is identical in construction to the previous model with the exception that the initial set of tetrahedra, depicted in Figure (2), consists of eight tetrahedra

$$\begin{array}{cccc} (1,2,4,5) & (1,2,5,6) & (1,2,6,7) & (1,2,4,7) \\ (1,3,4,5) & (1,3,5,6) & (1,3,6,7) & (1,3,4,7) \end{array}$$

In this model the common vertex is (1) and it is dragged forward and backward in time just as in the previous model.

2.3. Model 3

This is a four dimensional hypercubic lattice containing 384 4-simplices. This model can be built from a template hypercube, consisting of 24 4-simplices, replicated once along each of the four dimensions of the lattice. The details are as follows.

A single hypercube can be defined recursively as follows. Starting with one leg $(0,1)$ apply the following rules

$$\begin{aligned} (a,b) &\mapsto (0a,0b,1b)+(0a,1a,1b) \\ (a,b,c) &\mapsto (0a,0b,0c,1c)+(0a,0b,1b,1c)+(0a,1a,1b,1c) \\ (a,b,c,d) &\mapsto (0a,0b,0c,0d,1d)+(0a,0b,0c,1c,1d)+(0a,0b,1b,1c,1d)+(0a,1a,1b,1c,1d) \end{aligned}$$

The notation ia means append i as the most significant bit in the binary representation of a. The first rule generates two triangles from each leg, the second, three tetrahedra from each triangle and the final, four 4-simplices from each tetrahedron. Thus from one leg we will obtain 24 4-simplices defined by 16 vertices.

The above rules naturally assign a binary number to each of the 16 vertices. The bits of the binary number can be thought of as coordinates of the vertex (ie. the vertex with label $(abcd)$ is located at the point with coordinates (a,b,c,d) and a,b,c,d are in the set $\{0,1\}$). Note that these coordinates have nothing to do with metric or the physical coordinates of the lattice. They are introduced solely as an aid in the construction of the lattice.

This hypercube was used as a template in constructing a 2x2x2x2 hypercubic lattice with the 81 vertices labelled in lexicographic order. For the vertex with coordinates (a,b,c,d), where a,b,c,d are drawn from the set $\{0,1,2\}$, the lexicographic label is defined by

$$\texttt{lex}(a,b,c,d) = d+3(c+3(b+3a))$$

If we take $(p,q,r,s,t)_i$ to be $(p+i,q+i,r+i,s+i,t+i)$ then the full set of 384 4-simplices of the 2x2x2x2 hypercubic lattice are given by

$$
\begin{array}{llll}
(0,\ \ 1,\ \ 4,13,40)_i & (0,\ \ 1,\ \ 4,31,40)_i & (0,\ \ 1,10,13,40)_i & (0,\ \ 1,10,37,40)_i \\
(0,\ \ 1,28,31,40)_i & (0,\ \ 1,28,37,40)_i & (0,\ \ 3,\ \ 4,13,40)_i & (0,\ \ 3,\ \ 4,31,40)_i \\
(0,\ \ 3,12,13,40)_i & (0,\ \ 3,12,39,40)_i & (0,\ \ 3,30,31,40)_i & (0,\ \ 3,30,39,40)_i \\
(0,\ \ 9,10,13,40)_i & (0,\ \ 9,10,37,40)_i & (0,\ \ 9,12,13,40)_i & (0,\ \ 9,12,39,40)_i \\
(0,\ \ 9,36,37,40)_i & (0,\ \ 9,36,39,40)_i & (0,27,28,31,40)_i & (0,27,28,37,40)_i \\
(0,27,30,31,40)_i & (0,27,30,39,40)_i & (0,27,36,37,40)_i & (0,27,36,39,40)_i
\end{array}
$$

for $0 \leq i \leq \texttt{lex}(1,1,1,1)$.

The artificial coordinates (a,b,c,d) on each vertex can now be replaced with those appropriate for the particular spacetime under consideration.

The coordinates for the vertices for the first two models models were chosen according to Table (1) while for the hypercubic lattice the vertex labeled $\texttt{lex}(a,b,c,d)$ was assigned the coordinates

$$x^\mu(a,b,c,d) = \frac{1}{4}\left(a\delta^\mu_1 + 2(b+1)\delta^\mu_2 + 2(c+1)\delta^\mu_3 + 2(d+1)\delta^\mu_4\right)$$

with $x^\mu_\star = x^\mu(1,1,1,1)$.

Four exact solutions of the vacuum Einstein equations were used in evaluating the residual errors.

2.4. Metric 1 : Schwarzschild

The metric is described, in isotropic coordinates, by

$$ds^2 = -f^2(x,y,z)dt^2 + g^2(x,y,z)(dx^2 + dy^2 + dz^2)$$

where

$$f(x,y,z) = \left(\frac{2r-m}{2r+m}\right), \qquad g(x,y,z) = \left(1 + \frac{m}{2r}\right)^2$$

and $r^2 = x^2 + y^2 + z^2$ and the mass was set at $m = 1$.

2.5. Metric 2 : Kasner

This solution is described, again in pseudo-Cartesian coordinates, by

$$ds^2 = -e^{2t}dt^2 + e^{4t/3}dx^2 + e^{4t/3}dy^2 + e^{-2t/3}dz^2$$

It was chosen because it is asymmetrical and thus provides a more demanding test than that given by the Schwarzschild metric.

2.6. Metric 3: Plane wave

This is one of the many exact plane wave solutions and is described by

$$ds^2 = -4dudv + dx^2 + dy^2 + \big((x^2 - y^2)\sin u - 2xy\cos u\big)\, du^2$$

with $u = (t - x)/2$, $v = (t + x)/2$.

2.7. Metric 4: Plane symmetric wave

A second gravitational wave solution is the plane symmetric wave in the form

$$ds^2 = \frac{1}{\sqrt{z}}\left(-dt^2 + dz^2\right) + z\left(dx^2 + dy^2\right)$$

2.8. Metric 5: Reference

If we believe that the Regge calculus is a consistent approximation to General Relativity then we could reasonably expect the residual error to behave differently for exact solutions of Einstein's equations than for non-solutions.

Thus we chose to include a metric which was clearly not a solution of the vacuum Einstein's equations, namely,

$$ds^2 = -f^2(x, y, z)dt^2 + (dx^2 + dy^2 + dz^2)$$

with $f(x, y, z)$ chosen as in the Schwarzschild metric. We will use this metric as a reference by which we will compare the residual errors for the first four metrics.

For each of the above metrics and for each of the models the geodesics were computed with a step length in the Runge-Kutta routine of $\Delta\lambda = 1/10$ and ten iterations of the Newton-Raphson scheme. These figures were arrived at by experimentation. Increasing the number of iterations or decreasing $\Delta\lambda$ seemed to have no significant effect on the residual errors. In evaluating the partial derivatives (2.1) the δh^μ's were chosen as the constant value $\max_\nu(x^\nu(\lambda = 1) - x^\nu(\lambda = 0))/50$. Again, this was arrived at by experimentation (almost identical results were obtained using $\Delta\lambda = 1/5$, five iterations in the Newton-Raphson scheme and $\delta h = \max_\nu(x^\nu(\lambda = 1) - x^\nu(\lambda = 0))/25$).

3. The lattice equations

We have computed the residual errors for three different set of lattice equations. The most familiar is that of the Regge calculus. The other two are variations which though they have a Regge feel to them are in fact distinct from the Regge calculus.

3.1. The Regge Calculus

The standard Regge equations are

$$0 = \sum_{j(i)} \theta_j \frac{\partial A_j}{\partial L_i^2} \tag{3.1.1}$$

where the sum includes all of the triangles attached to a leg. There is one such equation for each leg in the lattice.

3.2. Miller's equations

Miller [4] obtained a set of equations by taking variations of the Hilbert action with respect to $g_{\mu\nu}$ rather than the L_{ij}. His equations are

$$0 = \sum_{i} \left(m\Delta x^\mu \Delta x^\nu\right)_i \sum_{j(i)} \theta_j \frac{\partial A_j}{\partial L_i^2} \tag{3.2.1}$$

where the outer sum is a sum over a set of legs, Δx^μ is the vector joining tip to tail of the leg and $0 \leq m \leq 1$ is a weighting assigned to the leg. The second sum is exactly the Regge equation for this leg (and thus Miller's equations are linear combinations of the Regge equations). The Δx^μ are computed with respect to a locally flat background metric (the error in doing so is of the order of a typical defect angle and these vary as $\mathcal{O}(\epsilon^2)$). There are two issues that remain, first, which legs should appear in the outer sum and, second, how should the weights m be calculated. Miller applied his equations only to a single hypercube within a 3x3x3x3 hypercubic lattice (ie. the hypercube built from the vertex $\texttt{lex}(1,1,1,1)$). He also defined the weights to be the fraction of the leg shared with adjacent hypercubes. Thus $m = 1$ for legs solely within the central hypercube and $m = 1/n$ for legs shared by the central and $n-1$ adjacent hypercubes.

We are unable to apply Miller's rules to both of our first and second models for it is not clear how to compute the Regge equations on the boundary legs nor how to assign a weight for such legs. Thus we shall choose to include, in the outer sum, only those legs attached to the central vertex of each model (ie. vertices 5,1 and $\texttt{lex}(1,1,1,1)$ for models 1,2 and 3). This is equivalent to setting $m = 1$ for legs directly attached to the central vertex and $m = 0$ for all other legs.

3.3. Brewin's equations

Brewin [6] has proposed the following lattice equations

$$0 = \sum_{j} \sum_{i(j)} \left(\Delta x^{\mu} \Delta x^{\nu}\right)_i \theta_j \frac{\partial A_j}{\partial L_i^2} \tag{3.3.1}$$

The outer sum contains only those triangles attached to the central vertex while the inner sum is a sum over the three legs of each triangle. There is one set of such equations for every vertex in the simplicial lattice. Note that these equations can be written as linear combinations of the Regge equations plus non-zero contributions from legs not attached to the central vertex.

These equations were derived by an integration of the vacuum Einstein equations directly on the lattice. The techniques used are very similar to those commonly used in finite element methods and amounts to nothing more than repeated integration by parts (for the details see [6]).

Note that if it were not for the weights and the surface terms, equation (3.2.1) would be identical to equation (3.3.1).

4. Results

We are primarily interested in the behaviour of the residual error τ as a function of the scale parameter ϵ, though we know that τ will also depend on the choice of metric, the lattice and the coordinates assigned to each vertex.

We shall begin by first forming some simple estimates for the behaviour of τ for small values ϵ.

First we note that if each bone in the lattice is non-null then τ is a smooth function of ϵ. This follows from the simple observation that a term in $\sum_i \theta_i \partial A_i / \partial L_j$ is singular only when its corresponding bone is null (in which case the defect is undefined). However by a careful choice of lattice and coordinates for each vertex we can always find an ϵ' such that for each $0 < \epsilon < \epsilon'$ each bone has a fixed signature and consequently each term in the residual error is a bounded function of ϵ.

We can also see that $\lim_{\epsilon\to 0} \theta_i = 0$ while $\lim_{\epsilon\to 0} \partial A_i/\partial L_j$ remains finite but non-zero. These deductions follow by inspection of the metric in (2.2). Clearly this metric is flat in the limit as $\epsilon \to 0$ and thus the defects vanish. Consequently we must have

$$\tau_i(g,\epsilon) = \epsilon^p Q(g,\epsilon)$$

where p is some undetermined (positive) number and $Q(g,\epsilon)$ is some undetermined function of ϵ. All that we need to know about $Q(q,\epsilon)$ is that there exists numbers $m(g)$ and $M(g)$

such that $0 < m(g) < |Q(g,\epsilon)| < M(g)$ for all ϵ in $(0,\epsilon')$. That $|Q(g,\epsilon)| < M(g)$ follows immediately from the fact that τ_i is bounded from above. That $0 < m(g)$ follows by choosing p such that $\lim_{\epsilon\to 0}\tau_i(g,\epsilon)\epsilon^{-p}$ remains finite and non-zero.

Note that p, m, M will depend not only upon the metric but also upon the topology of the lattice and the coordinates assigned to each vertex. They are however independent of ϵ.

Consider now a second metric (eg. the reference metric) for which we have

$$\tau_i(g',\epsilon) = \epsilon^q Q(g',\epsilon)$$

in the interval $0 < \epsilon < \epsilon''$ and q is some number (which may differ from p).

Our aim is to compare the residual errors for a pair of metrics, only one of which is a solution of Einstein's equations. Thus we are not particularly interested in the actual values of p and q but rather the value of $p - q$ (though later we will argue that $q = 2$ for all choices of g').

Thus consider

$$\frac{\tau_i(g,\epsilon)}{\tau_i(g',\epsilon)} = \epsilon^{p-q}\left(\frac{Q(g,\epsilon)}{Q(g',\epsilon)}\right)$$

for $0 < \epsilon < \epsilon^\star$ and $\epsilon^\star = \min(\epsilon',\epsilon'')$. Since $0 < m(g')$, the ratio $Q(g,\epsilon)/Q(g',\epsilon)$ is bounded. Thus[†] not only do we have $\tau_i(g,\epsilon) = \mathcal{O}(\epsilon^p)$ and $\tau_i(g',\epsilon) = \mathcal{O}(\epsilon^q)$ but we also have $\tau_i(g,\epsilon)/\tau_i(g',\epsilon) = \mathcal{O}(\epsilon^{p-q})$.

Note that these estimates are valid only when the signature of each bone is independent of ϵ for $0 < \epsilon < \epsilon^\star$. In every one of our cases studies this condition was met with $\epsilon^\star = 0.5$.

For each model and for each metric the **effective residual errors** were defined as

$$\eta_2(g,g') = \left(\frac{\sum_i \tau_i^2(g,\epsilon)}{\sum_i \tau_i^2(g',\epsilon)}\right)^{1/2}$$

where g' denotes the reference metric and the sums include every Regge equation in the computational patch.

If the Regge calculus is a consistent approximation to General Relativity then we should see $\eta_2(g,g') = \mathcal{O}(\epsilon^r)$ with $r > 0$. On the other hand if we observe $\eta_2(g,g') = \mathcal{O}(1)$ then we have some explaining to do for in that case we are unable, by these means, to distinguish between metrics which are solutions of Einstein's equations from those that are not.

The results for each of the numerical experiments are displayed in Figures (4–12). However, their interpretation is not completely clear cut. In some cases, such as in Brewin's and Miller's equations with Model 3, see Figures (9–12), we can see that effective residual errors for the exact solutions (metrics 1-4) vary as $\mathcal{O}(\epsilon^2)$ compared to the $\mathcal{O}(1)$ for the reference

† By $\tau = \mathcal{O}(\epsilon^p)$ we mean that there exists a constant K and $\epsilon^\star$ such that $|\tau| \leq K\epsilon^p$ whenever $0 < \epsilon < \epsilon^\star$, see [11]

metric. This is what we would expect for a consistent discrete approximation to Einstein's equations. Yet for other cases, such as for those of Model 1, see Figures (4,7,10), we see no distinction between the effective residual errors for the five metrics.

The worst results were found for Model 1 where the effective residual errors varied as $\mathcal{O}(1)$ for every metric and for each set of lattice equations. Models 2 and 3 displayed some degree of success but this may be attributed to their specialised construction (their legs were aligned to the coordinates axes of the continuum metric and the vertices were symmetrically placed relative to the central vertex).

To test this view we decided to repeat the calculations after introducing small fluctuations in the coordinates of the vertices

$$x^\mu_i \leftarrow x^\mu_i + 0.05(\omega_i - 0.5)x^\mu_i$$

where $0 \leq \omega_i \leq 1$ was a random number. By making these small changes we believe that we are using a simplicial lattice much closer to what we can expect in a generic application of the Regge calculus (or some other set of equations). Though these are, however, only minor changes.

We then found, without exception, that the effective residual errors varied as $\mathcal{O}(1)$. Thus we conclude that, for a generic simplicial lattice, the residual errors in each of the discrete equations (3.1.1–3.3.1) can not be used to distinguish between metrics which are solutions of Einstein's equations from those that are not.

5. Discussion

It would seem from the above that we are forced to accept one of two options : either that the Regge equations are not a consistent discretisation of Einstein's equations or that it is incorrect to use the residual errors of the Regge equations as a criteria in this context.

A common objection to this line of reasoning is that the negative result, that the effective residual varied as $\mathcal{O}(1)$, is a consequence of choosing either an inappropriate lattice, reference metric or vertex coordinates, and that with a better choice one might observe $\mathcal{O}(\epsilon^2)$ variation in the effective residual. We can, however, quickly exclude the option of changing reference metrics in the hope of improving the convergence. To see this recall that

$$\sum_i \theta_i A_i = \int_M \epsilon^2 R(x^\mu_\star + \epsilon(x^\mu - x^\mu_\star))\, dV$$

where the sum includes all bones inside the computational cell M and $\epsilon^2 R$ is the scalar curvature of the conformal metric (2.2). Over the patch M the curvature is bounded and the limits of integration do not depend on ϵ, thus

$$\sum_i \theta_i A_i = \mathcal{O}(\epsilon^2)$$

and as both A_i and L_j^2 are $\mathcal{O}(1)$ we find that

$$\sum_i \theta_i \frac{\partial A_i}{\partial L_j^2} = \mathcal{O}(\epsilon^2)$$

That is, for any choice of reference metric, we can expect $\tau_i(g', \epsilon) = \mathcal{O}(\epsilon^2)$ (this same statement has been made by Miller, see [4]) and consequently the effective residual $\eta_2(g, g')$ is independent of the choice of the reference metric.

Thus improving the convergence of η_2 for a given solution of Einstein's equations can only be achieved by changing the structure of the lattice. This has already been demonstrated with Models 1 and 3 for the Kasner metric where the convergence was seen to be $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon^2)$ respectively. However the real point is that there do exist cases where the effective residual does vary as $\mathcal{O}(1)$. In this specific case, with this lattice, metric etc. the effective residual makes no distinction between this solution of Einstein's equations and the reference metric. How can we explain this behaviour? Or do we have to accept that there are some classes of lattices for which the method of effective residuals is inappropriate? If so, then how do we characterise such lattices? We know of no way to do so and that is why we have included the second of our two options in the opening paragraph of this section.

If we reject the use of residual errors then we must justify why we can reject such a useful technique and furthermore we would be required to propose some other suitable criteria. One could well argue that what really matters is the convergence (or otherwise) of the solutions of the Regge equations to solutions of Einstein's equations. If we find convergence to any desired solution of Einstein's equations then the above behaviour of the residual errors just becomes a curious fact and the Regge calculus lives to fight another day.

Thus it seems reasonable to speculate on how one might achieve convergence at the level of solutions but non-convergence at the level of the field equations.

Consider, as a toy example, a second order differential equation

$$0 = \mathcal{L}(y)$$

for some function $y(x)$. Now construct a new function

$$\widetilde{y}_\epsilon(x) = y(x) + \epsilon^2 f(x/\epsilon)$$

where ϵ is arbitrary and $f(x)$ is any smooth bounded function of x. The function $\widetilde{y}_\epsilon$ will be the solution of some other second order equation

$$0 = \widetilde{\mathcal{L}}_\epsilon(\widetilde{y}_\epsilon) \tag{5.1}$$

Then it is easy to see that

$$|y - \widetilde{y}| = \mathcal{O}(\epsilon^2)$$

while

$$|\mathcal{L}(y) - \widetilde{\mathcal{L}}_\epsilon(y)| = \mathcal{O}(1)$$

where $|\cdots|$ is a point norm.

Thus the residual error is of $\mathcal{O}(1)$ yet the solutions are second order convergent. Thus in this toy model we can see that measuring the residual error alone is not sufficient to declare the discrete scheme to be invalid.

Even if we accept this, there will be some strange behaviour in the equations and their solutions. The term $\epsilon^2 f(x/\epsilon)$ corresponds to some irregular wave (since $f(x)$ is bounded) on the solution. The amplitude will decrease with ϵ but the frequency will increase. Thus this behaviour could be spotted by the appearance of a high frequency signal in the solution. Recent results by Gentle and Miller [3] displayed exactly this behaviour though it is not yet known if it is related to the mechanism described here. In the field equations this term $\epsilon^2 f(x/\epsilon)$ appears as a term of the form $f''(x/\epsilon)$. This also represents a high frequency term. Note that there is no well defined limit for this term as $\epsilon \to 0$ and thus we obtain a differential equation which does not have a well defined form.

If we are prepared to accept this mechanism then we must somehow explain why it has not been observed in previous applications of the Regge calculus.

All previous calculations with the Regge calculus (with the exception of the perihelion calculations of Brewin [7]) have been consistent with what one would expect from the Einstein equations. However, without exception, every numerical spacetime constructed to date using the Regge calculus has possessed a high degree of symmetry and those symmetries have been explicitly included in the simplicial lattice. Thus it is arguable that the Regge equations, for those spacetimes, were of such a simple form that they had no option but to converge to the correct Einstein equations. Indeed both the Kasner (Lewis [8]) and Schwarzschild (Wong [9]) spacetimes have been successfully constructed from the Regge calculus, a result which could not be expected nor discounted from the results presented here. One difference between their work and ours is their choice of lattice. Where we chose simplices they chose blocks specifically designed to reflect the symmetries of their target spacetimes. This might be significant if there is a non-uniqueness in the limiting form of the Regge equations. This is a real possibility when one considers the infinite choices one has in sub-dividing spacetimes into 4-simplices. It is also conceivable that for a given sequence of sub-divisions, the nature by which the leg-lengths are adjusted as the continuum is approached may have a bearing on the limiting form of the equations (eg. how are limiting spacetimes with a fractal-like structure avoided?).

One of the common arguments used to support the view that the Regge calculus should be a valid approximation to Einstein's theory is that both theories are derived from the same Hilbert action. However this similarity may not be as strong as it seems. The important point is that the configuration space for simplicial metrics is larger than that for smooth metrics (ie. metrics with bounded curvatures). To see this notice that every smooth metric can be arbitrarily approximated by a sequence of discrete metrics yet it is not possible to do the converse, to approximate a given discrete metric by a sequence of metrics with bounded curvatures. Thus the Regge equations arise from variations of the metric over a wider class than used in deriving the Einstein equations. Consequently one can not expect the two sets of equations to agree.

However, one can expect that the Einstein equations would arise as some linear combination of the Regge equations (arising from the constraints imposed in the reduction of the configuration space). In fact we have already seen linear combinations of the Regge equations arising naturally in both Brewin's and Miller's alternative theories (equations (3.3.1) and (3.2.1)). This averaging process may well also wash out the high frequency term $f''(x/\epsilon)$ noted above.

A simple example of this is given by the action

$$I(\theta,\phi) = \int \left(\theta_x^2 + \phi_y^2\right)\, dxdy$$

where θ and ϕ are arbitrary smooth functions of (x, y). The configuration space is the set of all pairs of smooth functions of two variables. Extremisation of I over this configuration space leads to two equations

$$0 = \theta_{xx} \qquad\qquad 0 = \phi_{yy}$$

while extremisation over the reduced configuration space where $\theta = \phi$ leads to the one equation

$$0 = \phi_{xx} + \phi_{yy}$$

Solutions of the original pair of equations are always solutions of the later equation. But the converse is not true, there exist many solutions of Laplace's equation which are not solutions of the previous pair. If this carries over to the Regge calculus then we can expect that Einstein's equations will be recovered as a linear combination of the Regge equations. Thus we can see that there may exist solutions of Einstein's equations which will not be solutions of the individual Regge equations though they may be solutions of an appropriately chosen linear combination of the Regge equations.

This observation addresses, in part, the claims of Barrett [20] that solutions of the linearized Regge equations should converge, in the sense of distributions, to solutions of the Einstein equations subject to a reasonable bound on the defects. However Barrett does not make any claims about the converse process, solutions of Einstein's equations being solutions of Regge's equations.

For these reasons we do not believe that the Regge equations (3.1.1) can be used to obtain accurate and consistent approximations to solutions of Einstein's equations for generic spacetimes. This claim is of course speculative and is open to criticism.

Let us now return to the question of finding a suitable criteria on which to judge the consistency of any proposed discrete set of equations. One option is to compare solutions of the discrete and exact equations. Yet, this is very difficult to do and is rather time consuming (we need to solve both set of equations – why do a job twice?).

We do however have an alternative in which we compute the residual error in the Einstein equations when evaluated on solutions of the discrete equations. This must surely be a stronger test than what we have been using. For if the residual error behaves appropriately (ie. vanishes at an appropriate rate) then there can be no doubt that the discrete equations do yield valid approximations to Einstein's equations and that the discrete solutions are correct.

How would we perform such a computation? It would require us to extract a smooth metric and in particular a point estimate of the curvature tensor (or Ricci tensor at worst) from the leg lengths of the lattice. To do so we will need to surrender the piecewise flat assumption (otherwise we do not get point estimates for the curvatures). Instead we could to try to fit a locally quadratic expansion of a smooth metric to the data on the lattice. The quadratic terms in this expansion should then give us our point estimate of the curvatures. This is very easily done if one uses Riemann normal coordinates and if one assumes the legs of the lattice to be short geodesic segments.

Notice also that in this approach there is actually no need for a separate set of discrete equations since we are able to use the Einstein equations directly on the lattice.

This work is currently in progress and we shall report on this in a later paper.

6. Acknowledgements

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7. References

[1] Regge, Tullio. (1961)
General Relativity without Coordinates.
Il Nuovo Cimento. Vol.XIX,No.3(1961) pp.558-571.

[**2**] Williams, R.M and Tuckey,P.A. (1992)
Regge Calculus : A bibliography and a brief review
Class.Q.Grav. Vol.9(1992) pp.1409-1422.

[**3**] Gentle, A.P. and Miller, W.A. (1997)
A fully (3+1)-d evolution of the Kasner cosmology using the Sorkin scheme
Submitted Class.Q.Grav.

[**4**] Miller,M.A. (1995)
Regge Calculus as a fourth order method in Numerical Relativity.
Class.Q.Grav. Vol.12(1995) pp.3037-3051.

[**5**] Richtmyer, R.D. and Morton, K.W. (1967)
Difference methods for initial value problems. 2nd edition.
Wiley-Interscience, New York, 1967.

[**6**] Brewin, L.C. (1997)
A Finite Element formulation of Simplicial General Relativity.
Preprint, Mathematics Department, Monash University.
Also available at `http://newton.maths.monash.edu.au:8000/preprints/felm-gr.ps.gz`

[**7**] Brewin, L.C. (1993)
Particle paths in a Schwarzschild spacetime via the Regge Calculus.
Class.Q.Grav. Vol.10(1993) pp.1803-1823.

[**8**] Lewis, S.M. (1972)
Two cosmological solutions of Regge Calculus.
Phys.Rev.D. Vol.25(1982) pp.306-312.

[**9**] Cheuk-yin Wong. (1971)
Application of Regge Calculus to the Schwarzschild and Reissner–Nordstrom geometries at the moment of time symmetry.
J.Math.Phys. Vol.12(1971) pp.70-78.

[**10**] Barrett, J.W. (1988)
A convergence result for linearised Regge Calculus.
Class.Q.Grav. Vol.5(1988) pp.1187-1192.

[**11**] Hardy, G.H. (1958)
A course of pure mathematics.
Cambridge University Press (1958).

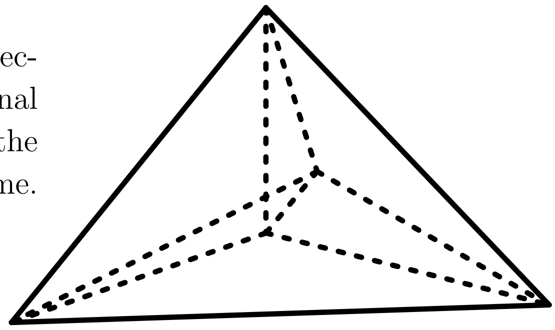
Coordinates for Model 1				
Vertex	(x^1)	(x^2)	(x^3)	(x^4)
1	0.500	0.500	0.500	0.500
2	0.500	1.000	0.500	0.500
3	0.500	0.500	1.000	0.500
4	0.500	0.500	0.500	1.000
5	0.500	0.625	0.625	0.625
$5^{\uparrow}$	1.000	0.625	0.625	0.625
$5^{\downarrow}$	0.000	0.625	0.625	0.625
$V_\star$	0.500	0.625	0.625	0.625

Table 1. Coordinates of the vertices of the various models.

Coordinates for Model 2				
Vertex	(x^1)	(x^2)	(x^3)	(x^4)
1	0.500	0.500	0.500	0.500
2	0.500	0.500	0.500	1.000
3	0.500	0.500	0.500	0.000
4	0.500	1.000	0.500	0.500
5	0.500	0.500	1.000	0.500
6	0.500	0.000	0.500	0.500
7	0.500	0.500	0.000	0.500
$1^{\uparrow}$	1.000	0.500	0.500	0.500
$1^{\downarrow}$	0.000	0.500	0.500	0.500
$V_\star$	0.500	0.500	0.500	0.500

Coordinates for Model 3				
Vertex	(x^1)	(x^2)	(x^3)	(x^4)
0–255	See text			
$V_\star$	0.250	1.000	1.000	1.000

Figure 1. The first model, a simple collection of 4 tetrahedra. The full 4-dimensional simplicial space is obtained by displacing the central vertex forward and backward in time.



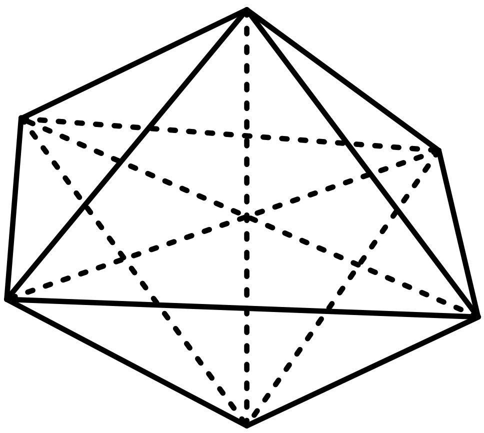
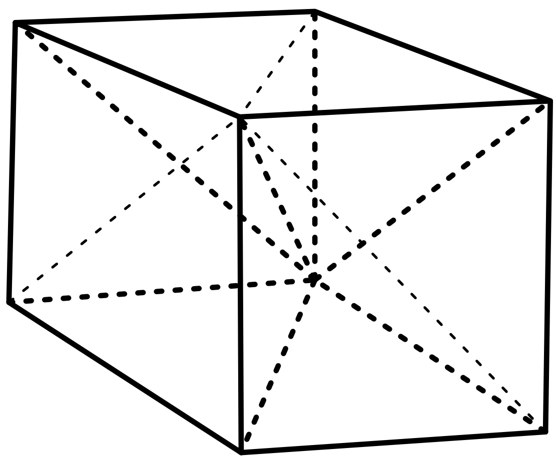


Figure 2. The second model. In this model the central vertex is also dragged forward and backward in time.

Figure 3. An example of a three dimensional hypercube. The heavy dotted lines are part of the template for this hypercube. The light dotted lines arise form templates of other hypercubes.



Effective Residual Errors, Model 1

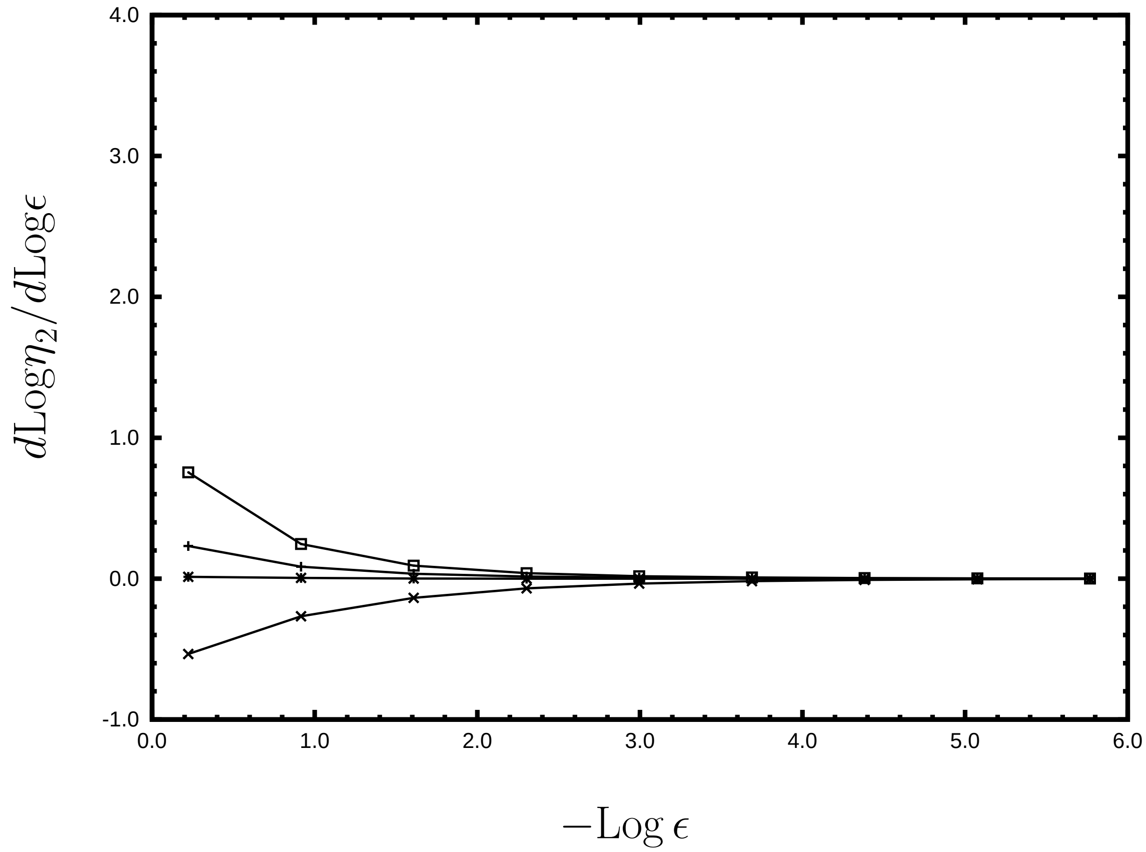


Figure 4. This graph displays $d\log\eta_2/d\log\epsilon$ as a function of $-\log\epsilon$ for the original Regge equations for Model 1. The squares correspond to the Schwarzschild spacetime, triangles to the Kasner solution, diamonds to the plane wave, stars to the symmetric wave. Note that for every metric the effective residual error varies as $\mathcal{O}(1)$. The derivatives were estimated using $d\eta/d\epsilon = (\eta_{i+1}-\eta_i)/(\epsilon_{i+1}-\epsilon_i)$.

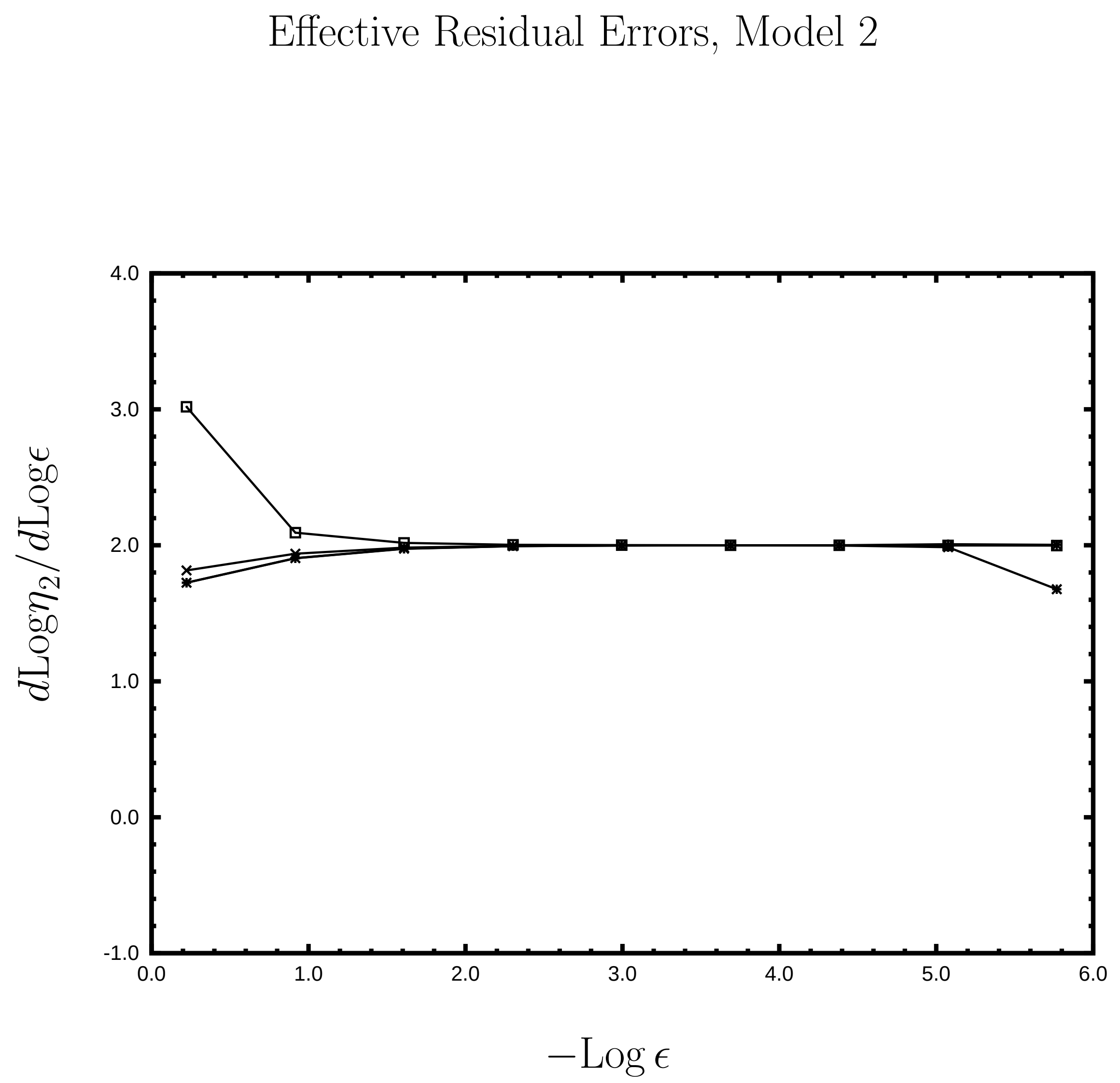


Figure 5. Results for the original Regge equations for Model 2. For the Kasner and both plane wave metrics the effective residual error varies as $\mathcal{O}(\epsilon^2)$ while for the Schwarzschild metric we only have $\mathcal{O}(1)$. The dip in the curves near $5 < -\text{Log}\epsilon < 6$ indicates that machine precision has been reached. This behaviour can also be seen in many of the other plots.

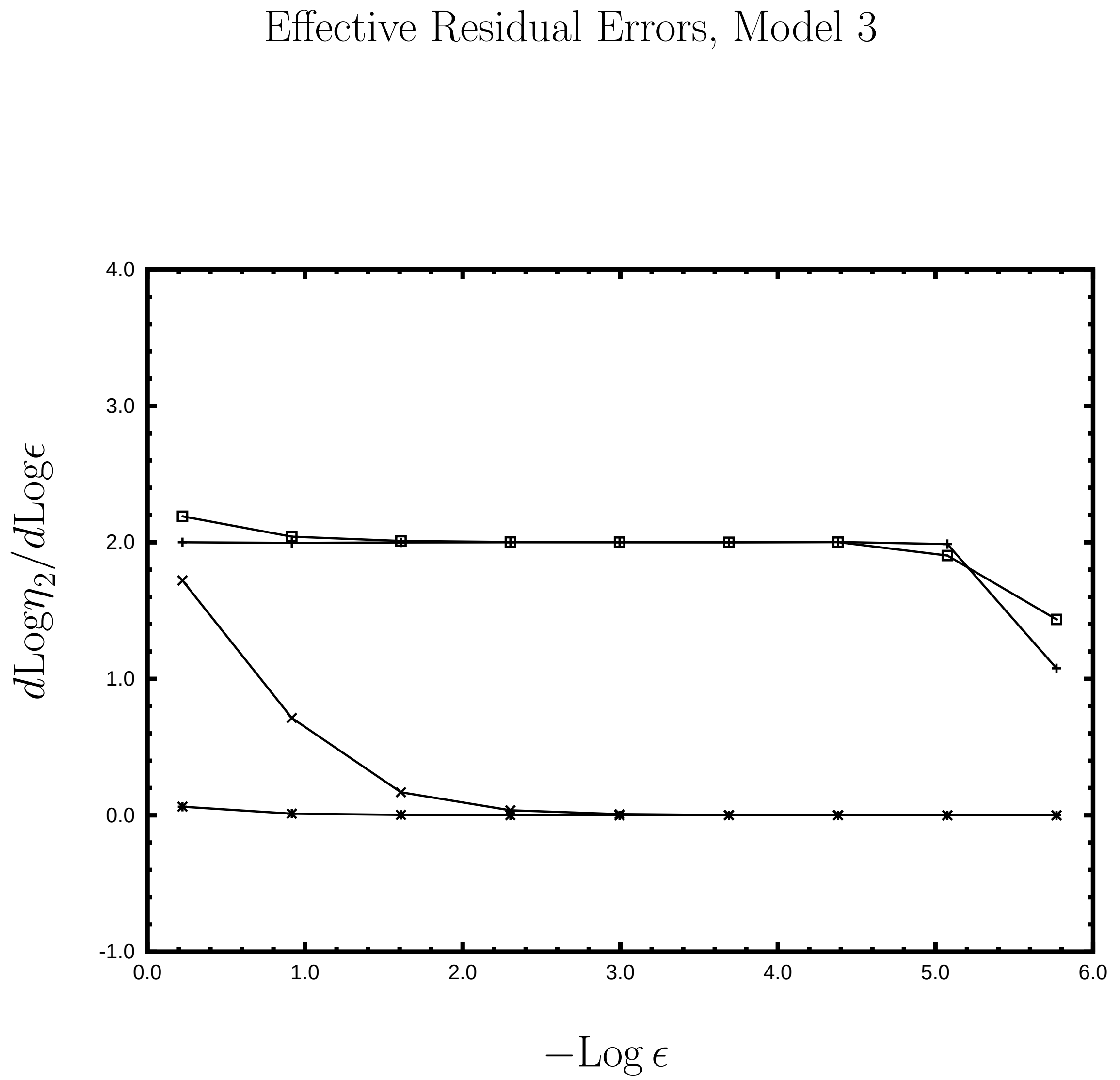


Figure 6. Results for the original Regge equations for Model 3. This displays an $\mathcal{O}(\epsilon^2)$ effective residual error for only the Kasner and plane wave metrics.

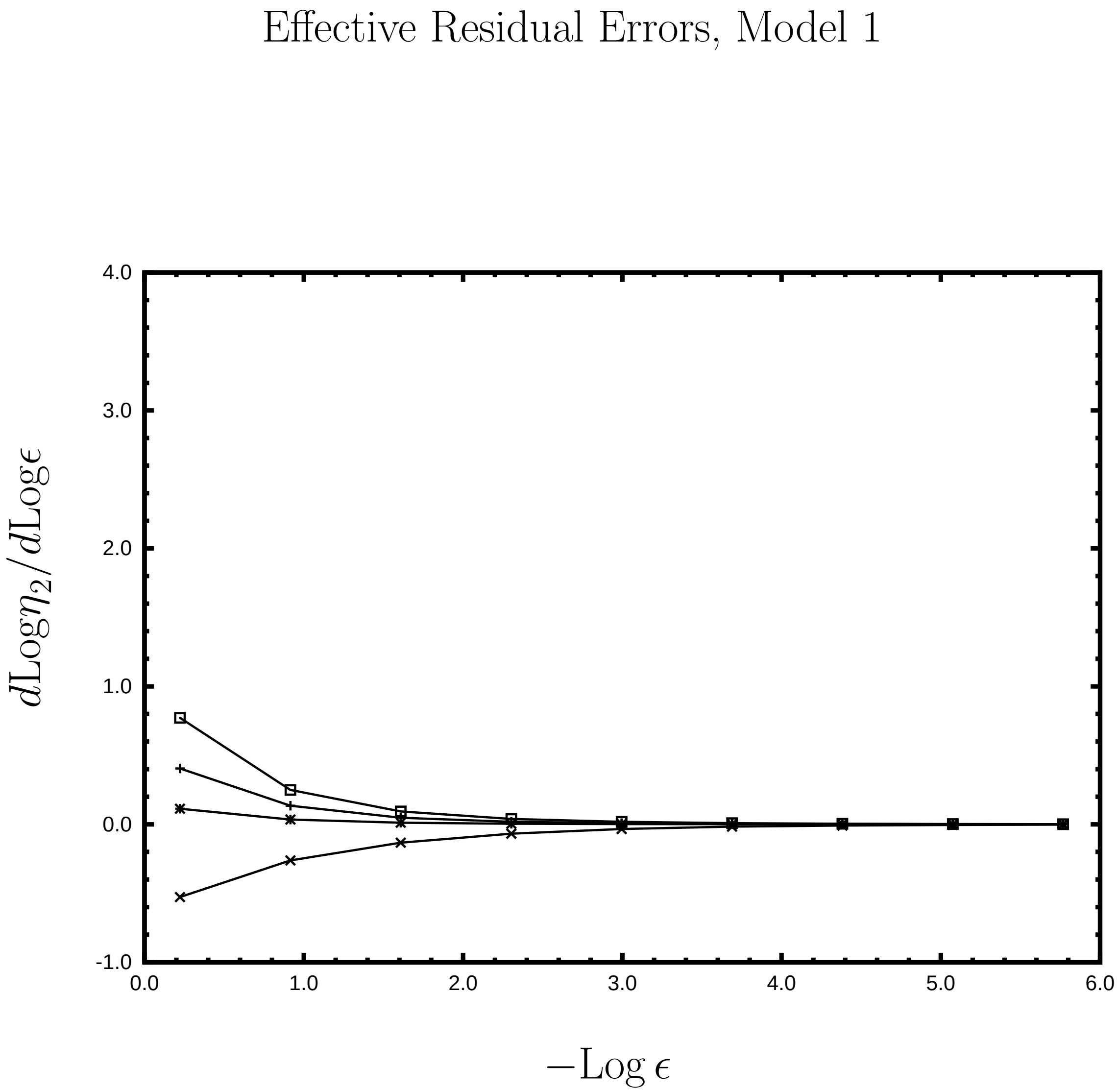


Figure 7. Results for the Brewin's equations for Model 1.

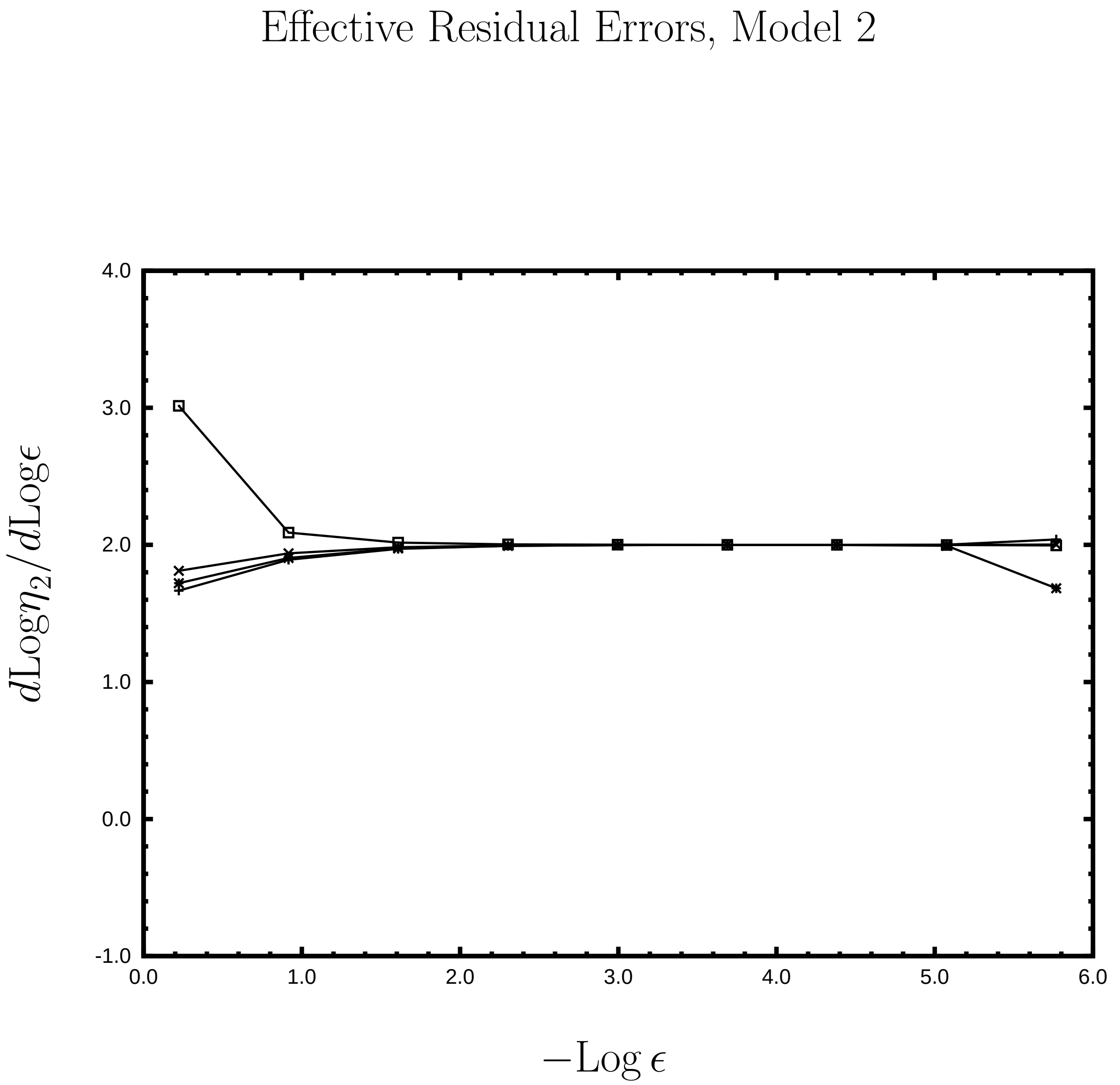


Figure 8. Results for the Brewin's equations for Model 2.

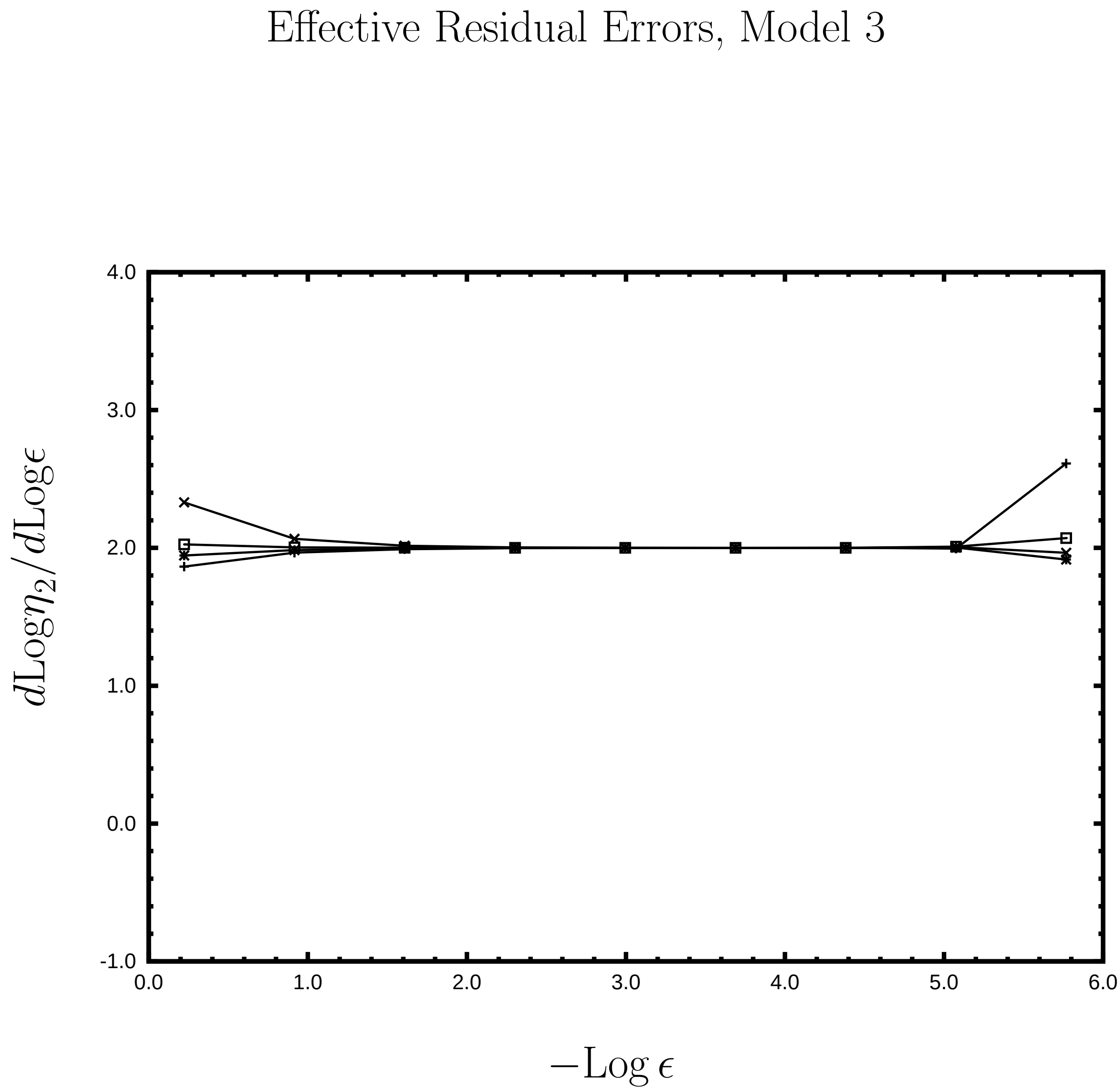


Figure 9. Results for the Brewin's equations for Model 3.

Effective Residual Errors, Model 1

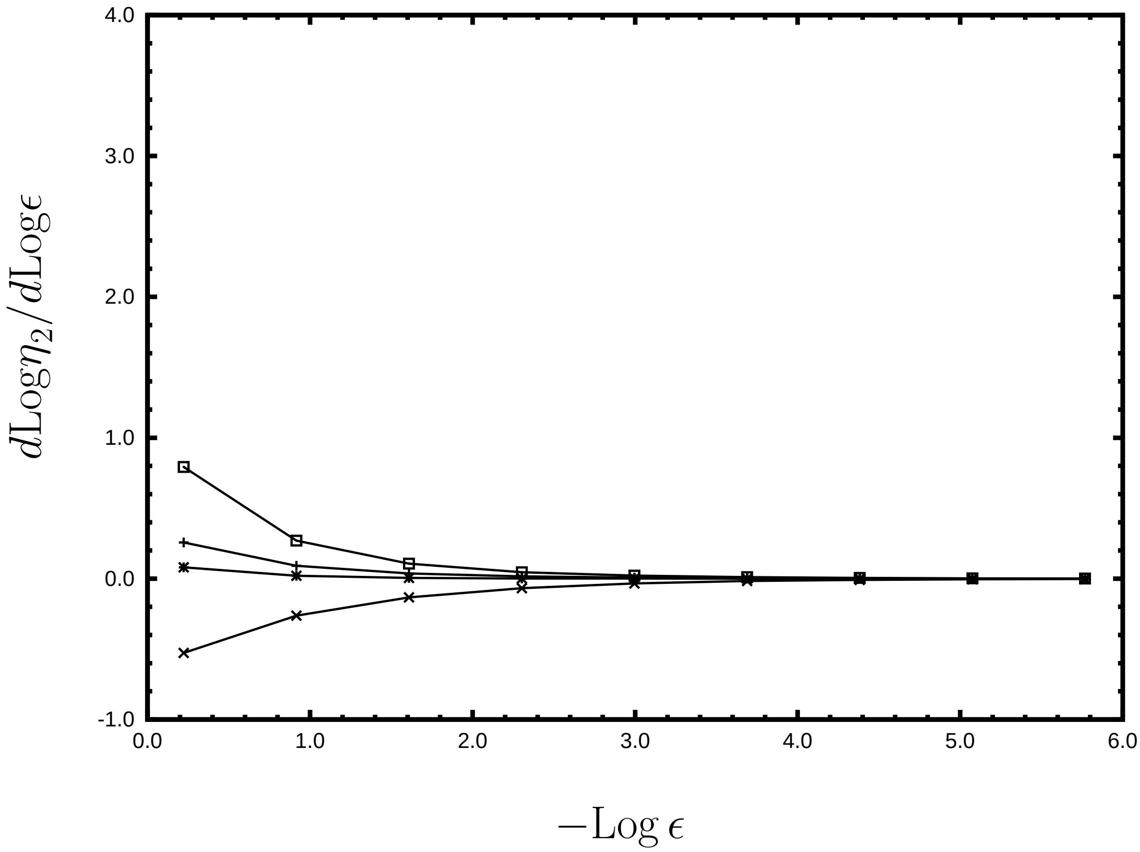


Figure 10. Results for Miller's equations (modified as per section (3.2)) for Model 1.

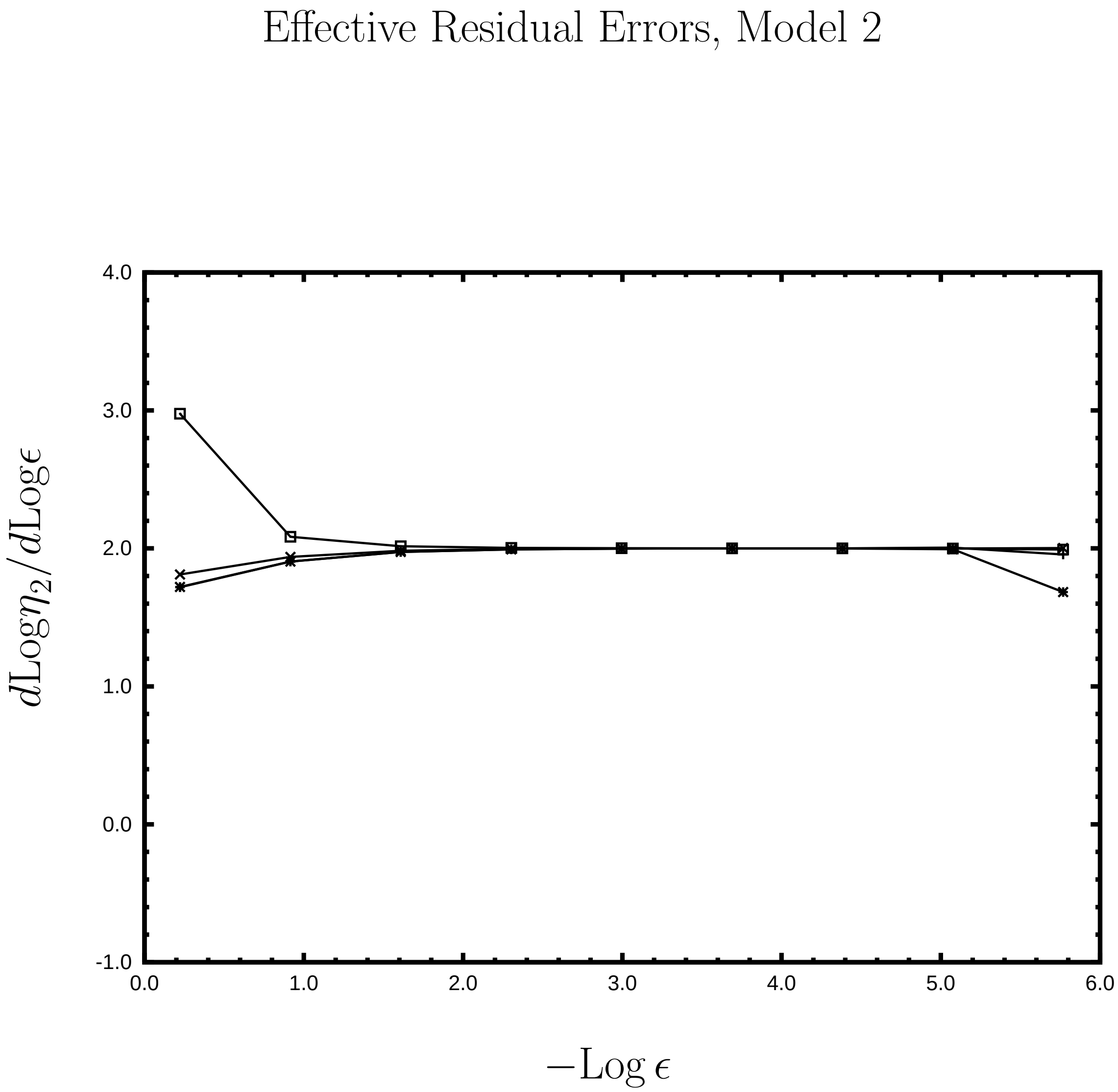


Figure 11. Results for Miller's equations (modified as per section (3.2)) for Model 2.

Effective Residual Errors, Model 3

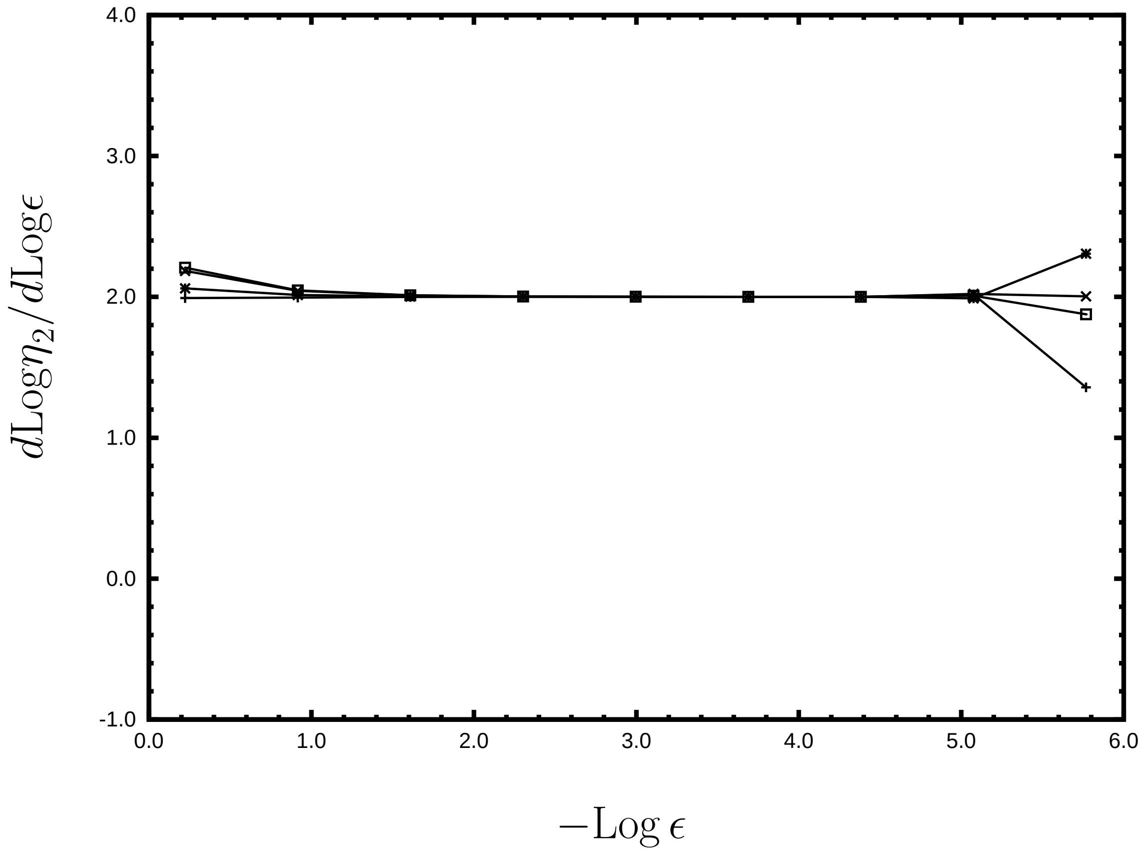


Figure 12. Results for Miller's equations for Model 3.