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On the convergence of Regge calculus to general relativity

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Abstract. Motivated by a recent study casting doubt on the correspondence between Regge calculus and general relativity in the continuum limit, we explore a mechanism by which the simplicial solutions can converge whilst the residual of the Regge equations evaluated on the continuum solutions does not. By directly constructing simplicial solutions for the Kasner cosmology we show that the oscillatory behaviour of the discrepancy between the Einstein and Regge solutions reconciles the apparent conflict between the results of Brewin and those of previous studies. We conclude that solutions of Regge calculus are, in general, expected to be second order accurate approximations to the corresponding continuum solutions.

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1. Introduction

Regge calculus [1] is a discrete theory of gravity which replaces the smoothly curved spacetime of general relativity with a lattice. The curvature of the lattice spacetime is concentrated entirely on the two-dimensional hinges of the four-dimensional lattice cells.

Regge calculus holds much promise for the numerical investigation of both classical and quantum gravity. Although the lattice approach appears well suited to numerical applications, progress in the field has been slow. Only recently have the first completely generic four-dimensional numerical simulations been performed [2, 3].

After the recent papers by Brewin [4] and M. Miller [5], and despite the proven track record of the Regge calculus, a lively debate arose [6] as to whether or not solutions of the Regge equations would converge to solutions of the Einstein equations in some suitable limit. Neither Brewin or M. Miller directly computed solutions of the Regge equations. Instead, they took the somewhat easier approach of evaluating the Regge equations on an exact solution of Einstein's equations. They did this using sophisticated interpolation schemes, based on geodesics, to map a range of Einstein solutions onto simplicial lattices. The residual of the resultant Regge equations, calculated using these interpolated lattice edge lengths, was then examined in the limit

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of very fine lattice discretisations. Brewin observed that the residual scaled as $\mathcal{O}(1)$ as the lattice was refined, and he inferred that solutions of the Regge equations, in generic spacetimes, would not converge to solutions of the Einstein equations in the limit of fine discretisations. M. Miller chose a slightly different measure for the residual of the Regge equations, but when expressed in terms of Brewin's measure, M. Miller's results are consistent with those obtained by Brewin. Despite this M. Miller does draw sharply different conclusions, based upon further calculations associated with averages of the Regge equations, namely that solutions of the standard Regge equations will converge as the fourth order in the lattice spacing to solutions of the Einstein equations. Similar behaviour, in which the residual converges only over averages of the equations, has been observed by Sorkin [7] in the context of massless scalar fields on a 2-dimensional simplicial space.

The situation became even more confused after the work of Gentle and W. Miller [2], who demonstrated explicit quadratic convergence of particular solutions of Regge calculus to a solution, the Kasner spacetime, of general relativity. This spacetime was one of the test cases used by Brewin and M. Miller. This seems most odd – we appear to have a convergent set of solutions from an apparently non-convergent set of equations. How can this be?

In this paper we explore a possible explanation for this behaviour, as proposed by Brewin [4], which is consistent with all previous numerical studies. It is important to note that the direct solution of the Regge equations has always generated solutions which converge to the corresponding solutions of general relativity. Although the observed rate of convergence is dependent upon the particular lattice and symmetry restrictions imposed, when the lattice construction allows the full expression of the gravitational degrees of freedom, second order convergence of the lattice solutions to the continuum has always been observed [3].

We begin with a more complete description of the problem in section 2, and then discuss a possible explanation in section 3. Finally, in section 4, we present numerical evidence to support our proposal.

2. Investigating the convergence of Regge calculus

There are various ways in which the convergence properties of a numerical technique can be explored. The most rigorous approach involves the direct comparison of the approximate equations and the full system, concentrating on the leading order discrepancies. The underlying assumption is that the solutions of the approximate equations will converge at the same rate as the approximate equations themselves converge to the original system.

The Regge and Einstein equations are too complex for such a direct approach to be beneficial. Not only are the equations inherently complex non-linear systems, but before such an analysis can proceed one must choose a lattice upon which to express the Regge equations, together with a coordinate system in which to write out the Einstein equations. Once these decisions have been made, we are faced with a more problematic choice: how are the two sets of equations to be compared? In general there are more Regge equations (one per lattice edge) than Einstein equations (ten per spacetime event).

There is no clear prescription for averaging the lattice equations to obtain the correct number of Einstein equations in the continuum limit. Brewin [4] and M. Miller [5] have both developed their own schemes for averaging the Regge equations. A

direct assault on the convergence properties of Regge calculus is therefore likely to be applicable only on a specific lattice, with a particular choice of averaging in the continuum limit. That is, the results would indicate that, in some averaged sense, the Regge equations converge to the Einstein equations. However, the result would not constitute a proof valid for all possible lattice choices.

Given these limitations, it is natural to ask if similar results can be obtained by directly solving the Regge equations and comparing the lattice solutions to corresponding Einstein spacetimes. Brewin and M. Miller have investigated the inverse problem: given a known (analytic) solution of general relativity, and interpolating that solution onto a lattice, what can be said about the convergence of the Regge equations?

The approach taken by Brewin and M. Miller was to introduce some discretisation process in the smooth manifold on which the Einstein equations are defined. This provides a way to map the smooth metric solutions to a lattice, giving a set of lattice edge lengths $\mathbf{L}_E$ derived from the continuum metric. Both authors used geodesic lengths to map continuum information onto the lattice spacetime. These new "continuum" lattice edge lengths do not in general satisfy the Regge equations, but we can evaluate the residual

$$r = |\mathbf{R}(\mathbf{L}_E)| \tag{1}$$

which is an indication of how well the interpolated Einstein solution satisfies the Regge equations. It is this residual, using an appropriately chosen norm, which both Brewin and M. Miller considered [4, 5].

Brewin observed that the residual of the simplicial equations remained roughly constant as the lattice was refined on a fixed region of spacetime. This result led Brewin to question the validity of Regge calculus, since any useful numerical scheme must converge to the underlying solution of the partial differential equations as the resolution is improved. M. Miller observed second order convergence of the residual, for any smooth metric, whether or not they were solutions of Einstein's equations. Had he used the same norm for the residual as used by Brewin we believe he too would have observed $\mathcal{O}(1)$ convergence of the residuals.

It is important to note that the experiments of Brewin and M. Miller do not directly evaluate the convergence of the numerical solutions. Rather, they investigate the convergence of the lattice equations to the Einstein equations. As the lattice is refined, it is reasonable to expect that the interpolated Einstein solution will satisfy the Regge equations increasingly accurately. This, however, is not what Brewin observed numerically.

3. A possible explanation

The observations of Brewin are particularly puzzling in light of the many previous applications of Regge calculus. Almost every numerical application of the method has displayed convergence towards the corresponding continuum solution, with most studies indicating that numerical Regge calculus is a second order accurate approximation to general relativity.

The edge lengths measured in the Regge lattice (L) and the corresponding interpolated Einstein edges (L_E, obtained by assigning geodesic lengths calculated in the continuum) are related as

$$L = L_E + \mathcal{O}(\delta^{p+1}) \tag{2}$$

with previous numerical studies suggesting that $p = 2$ (see Gentle [3] for case studies and a general review). Throughout this paper we assume that δ is a typical length scale in the lattice, and noting that the edges themselves are of this magnitude, we say that edges which satisfy equation (2) are p-order accurate approximations to the continuum solution.

Brewin [4] has proposed a mechanism whereby the solutions of the Regge equations converge to the corresponding Einstein solutions, while at the same time the residual of the Regge equations evaluated on the interpolated Einstein lattice edges does not converge. The key to Brewin's proposal is allowing the functional form of the error terms (the discrepancy between the Einstein and Regge solutions) to depend on the discretisation scale δ. This can be clearly seen from a toy model.

Suppose $\mathcal{L}$ is a second order differential operator, and $y(x)$ is a solution of the equation

$$\mathcal{L}y = 0. \tag{3}$$

Brewin considers the function

$$\tilde{y}_\delta(x) = y(x) + \delta^2 f(x/\delta) \tag{4}$$

for some arbitrary scalar δ and an arbitrary, though bounded, function $f(x)$. Considering the difference between the solutions we find that

$$|y(x) - \tilde{y}_\delta(x)| = \mathcal{O}(\delta^2), \tag{5}$$

indicating that the solutions differ only by "second order" terms. Noting that $\tilde{y}_\delta(x)$ is a solution of some other related equation $\tilde{\mathcal{L}}_\delta$,

$$\tilde{\mathcal{L}}_\delta \tilde{y}_\delta = 0, \tag{6}$$

where $\tilde{\mathcal{L}}_\delta$ is also a second order differential operator (though different from $\mathcal{L}$), we find that the "residual" of $\tilde{y}_\delta(x)$ with respect to the original operator is

$$|\mathcal{L}\,\tilde{y}_\delta| = \mathcal{O}(1). \tag{7}$$

This toy model embodies precisely the properties observed by Brewin – second order convergence of the solutions [3], with no corresponding convergence observed in the Regge equations when they are evaluated on exact solutions interpolated from the continuum [4]. The discrepancy between the two solutions in this toy model is seen to be a wave-like disturbance with frequency proportional to $1/\delta$.

In the continuum limit it is reasonable to expect that the discrete Regge equations (or a weighted average over them) approach a system of differential equations. Moreover, consideration of the Einstein equations leads us to expect that the limiting form of the Regge equations is a set of second order non-linear equations. Furthermore, previous numerical experiments suggest that Regge calculus is a second order method; we expect $p = 2$. The toy model considered above leads us to expect that if the lattice solutions differ from the continuum solutions by terms with frequencies proportional to $1/\delta$, it is not unreasonable that the residuals of the Regge equations remain roughly constant as the resolution is improved.

This is precisely the behaviour observed by Brewin [4]; the solutions converge even though the residual of the equations do not. This explanation relies on the existence of high frequency, low amplitude waves in the simplicial solutions; a possibility not ruled out by any of the largely low-resolution applications of Regge calculus to date. In fact, one study hints at the existence of precisely this type of wave-like structure in the Regge solutions [2]. In the next section we construct high resolution solutions of the Regge equations in order to gain insight into the fine-scale behaviour of the lattice solutions.

4. Numerical solution of the Regge equations

In this section we solve the Regge equations for the vacuum Kasner cosmology using a $(3+1)$-dimensional formulation of Regge calculus described elsewhere [2]. The initial value problem is solved and the lattice is evolved subject to simplicial lapse and shift conditions.

The Kasner cosmology is a homogeneous anisotropic exact solution of the vacuum Einstein equations, and is the prototype for generic velocity dominated cosmological singularities. It also provides a model for epochs in the mixmaster cosmology between bounces. The metric may be written in the form

$$ds^2 = -dt^2 + t^{p_1}dx^2 + t^{p_2}dy^2 + t^{p_3}dz^2, \tag{8}$$

where the field equations reduce to a pair of algebraic constraints,

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1. \tag{9}$$

The Kasner exponents (p_1, p_2, p_3) are a one-parameter set, and the solutions include the flat-spacetime case $(0, 0, 1)$.

The previous application of Regge calculus to the Kasner cosmology was viewed as a benchmark of the numerical technique, and as such the convergence of the solutions was an important issue. It was found that the lattice solution converged to the continuum as the second power of the typical lattice spacing [2]. This earlier study used a fixed lattice resolution, containing 128 vertices arranged in two offset cubic grids each consisting of $4 \times 4 \times 4$ vertices. Reducing the overall scale of the lattice, combined with the T^3 topology of the Kasner solution, allowed the convergence rate to be estimated.

In this paper we directly subdivide the lattice whilst fixing the size of the spatial region. This is a more general and demanding convergence test, as any long wavelength modes excluded in the previous study are now resolvable. Since the major issue is convergence we refine the lattice in only one dimension, but reduce the scale of the remaining spatial axes to avoid the creation of long skinny triangles. Such long and skinny triangular elements are known to cause instabilities in finite element calculations.

This effectively "one-dimensional" model allows convergence analysis to be conducted whilst containing the computational scale of the problem. All calculations are performed on a lattice containing two offset cubic grids which are fully subdivided into triangles, tetrahedra and four-simplices. Each cubic grid contains $2^n \times 4 \times 4$ vertices ($N \equiv 2^n = 4, 8, \ldots, 1024$), with the x-axis having a fixed length of $X = 10$, and the remaining axes scaled to compensate; $Y = Z = 40/2^n$. This ensures that the spatial edges surrounding a vertex are all of the same order of magnitude. The typical spatial resolution in each model is thus $\delta \approx 10/2^n$.

The full four-dimensional two-slice initial value problem is solved at $t = 1$, after which the lattice is evolved to $t = 2$ using the $(3+1)$-dimensional Sorkin evolution scheme and "geodesic slicing" conditions; that is, with unit lapse and zero shift. The time step is chosen for each resolution to give a Courant factor of $dt/l \approx 0.2$; for $N = 512$ we chose $dt = 3.9 \times 10^{-3}$. However, the main results of this paper were found to be largely insensitive to the choice of timestep, provided it satisfied the Courant condition. For full details of the evolution and initial value algorithms, see Gentle and W. Miller [2]. The results of the Regge evolution are compared with the continuum solution by performing a least-squares fit of the function

$$L(t) = L_0 t^{p_r} \tag{10}$$

to the time evolution of each class of the axis-aligned spatial edges, L_x, L_y and L_z. In agreement with previous studies, the lattice solution is found to remain spatially homogeneous throughout the evolution to within roundoff error. The typical standard deviation of the edges also remained below the limit of numerical accuracy.

Figure 1 displays the fractional difference between the early evolution of the Regge cosmology and the exact solution for the axisymmetric case with $p_1 = p_2 = 2/3, p_3 = -1/3$. Although the average magnitude of the discrepancy is small, there is a clear oscillation superimposed upon a general trend. The figure displays data for three lattices consisting of $128 \times 4 \times 4$, $256 \times 4 \times 4$ and $512 \times 4 \times 4$ vertices.

An estimate of the rate at which the Regge solution converges to the continuum is given in figure 2. After performing a least squares fit of equation (10) to the simplicial solution in the region $t = [1, 2]$, we plot the fractional difference between the fitted power p_r and the analytic value p_c for each spatial axis. It is clear that this power law fit approaches the continuum solution as the second power of the lattice spacing. Figure 3 shows the scaled error in the least squares fit, giving an indication of how well the power law (10) describes the Regge solution. Again, we find second order convergence.

The data represented in figure 3 contains additional information. It is clear from figure 1 that there is a strong wave-like oscillation superimposed upon the general error curve. Fitting equation (10) to such data will draw out the averaged solution, while the error in the fit will reflect the magnitude of any discrepancies from that average behaviour. Thus we see that both the oscillations and the underlying trend in figure 1 converge to zero as at least the second power of the grid spacing.

In light of the discussion in the previous section, the frequency of the oscillations evident in figure 1 is also of interest. It is clear from the figure that the frequency increases with the lattice resolution. Figure 4 shows this in a more quantitative manner, displaying the Fourier transform of a small segment of the fractional error. It is apparent that the frequency of the waves varies linearly with the number of vertices, or alternatively, that the wavelength of the oscillations is proportional to $1/N$.

Together, these $(3 + 1)$-dimensional convergence results show that the numerical Regge solution is a second order accurate approximation to the corresponding solution of the Einstein equations. Moreover, the leading order difference between the two solutions contains high frequency, bounded oscillations. We have shown that the frequency of these waves is proportional to N (or $1/\delta$), while their magnitude reduces as $1/N^2$ (δ^2). This is precisely the variation postulated by Brewin [4], and strongly supports his explanation of the apparent non-convergence of the Regge equations.

Finally, we note that the Kasner solution displays only temporal oscillations, with the spatial three-geometries remaining homogeneous to high accuracy throughout the evolution. This is likely a result of the high degree of symmetry of the Kasner spacetime, in which the constant time spatial hypersurfaces are flat. In a more general setting we would expect to see similar high frequency, bounded oscillations in both space and time.

5. Conclusion

We have carried out a rigorous convergence study of a particular solution to the Regge equations, and shown that the solutions do indeed converge to the corresponding Einstein solutions in the limit of very fine discretisations.

Following a suggestion of Brewin we examined the behaviour of the error terms, and found convincing evidence for the existence of high frequency, low amplitude oscillations superimposed upon the continuum solution. Brewin previously suggested that waves of precisely this form in the simplicial solution could explain the lack of convergence observed in the residual of the Regge equations when they are evaluated on the interpolated continuum solution.

Together these results suggest that solutions of the Regge equations are generally second order accurate approximations to the corresponding Einstein spacetimes, with the discrepancy between the two solutions consisting of high frequency, low amplitude waves. Although these waves prevent the residual of the Regge equations converging to zero when evaluated on interpolated Einstein solutions [4], they do not affect the overall second order accuracy of the simplicial solutions.

We expect that all generic Regge simulations in vacuum will produce similar results. That is, all simulations will contain high frequency oscillations, linked to the inverse of the discretisation scale, which reduce in magnitude as the second power of that scale. These oscillations do not appear to induce instabilities in the numerical evolution of simplicial lattices. Indeed, previous studies have indicated that the oscillations gradually decay as the evolution proceeds [2]. However, M. Miller [8] has reported instabilities arising in linearised Regge calculus on asymmetrical grids.

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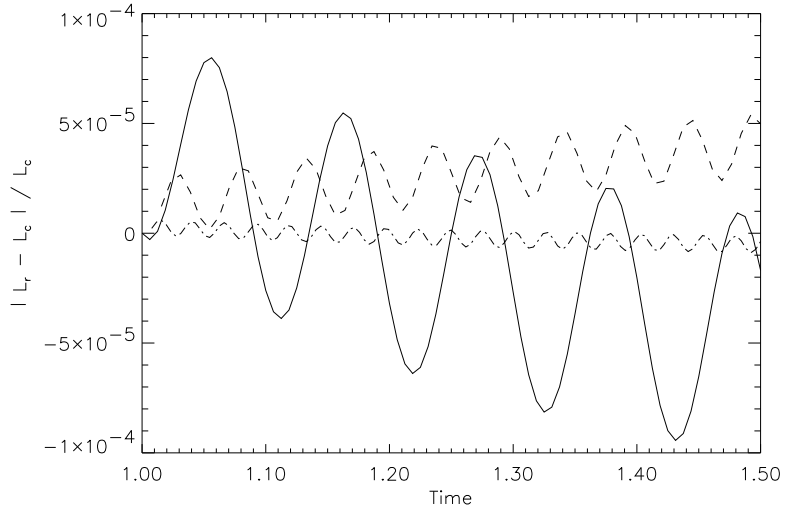

Figure 1. The fractional error $\delta L/L$ of a typical edge in the Regge lattice as a function of time. Results are shown for spatial edges aligned with the x-axis, using lattice resolutions of $128 \times 4 \times 4$ (——), $256 \times 4 \times 4$ (– – –), and $512 \times 4 \times 4$ (— · —) vertices. In all cases the total length of the lattice along the x-axis is 10.0 units. The magnitude of the observed oscillations are proportional to $1/N^2$, while the frequency increases with N.

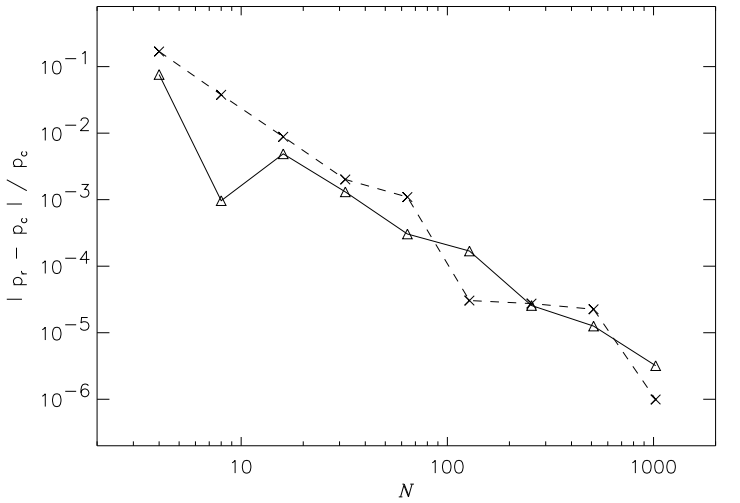

Figure 2. A least squares fit of the function $L_0 t^{p_r}$ (equation 10) is performed on the edge lengths at different resolutions. We plot the fractional error in the fitted power p_r as a function of the number of vertices. The fractional error in the fit along the x and y-axes ($\triangle$-points; $p \approx 2/3$) are indistinguishable; the z-axis fit ($\times$-points; $p \approx -1/3$) follow the power-law behaviour. The fractional error in the average Regge solution can thus be seen to vary as $1/N^2$.

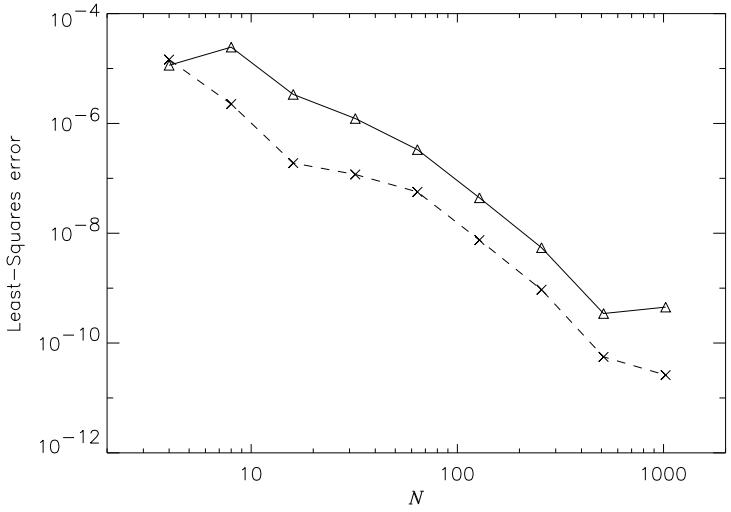

Figure 3. The error in the least squares fit of equation (10) to the mean edge lengths is shown as a function of the number of vertices. The results for the x and y-axes are again indistinguishable ($\triangle$), with the z-axis showing the same trend ($\times$). This is a measure of both the accuracy of the functional form given in equation (10) and the amplitude of the waves shown in figure 1. We again find that the error measure converges to zero as $1/N^2$, indicating that the wave amplitude reduces as at least $1/N^2$.

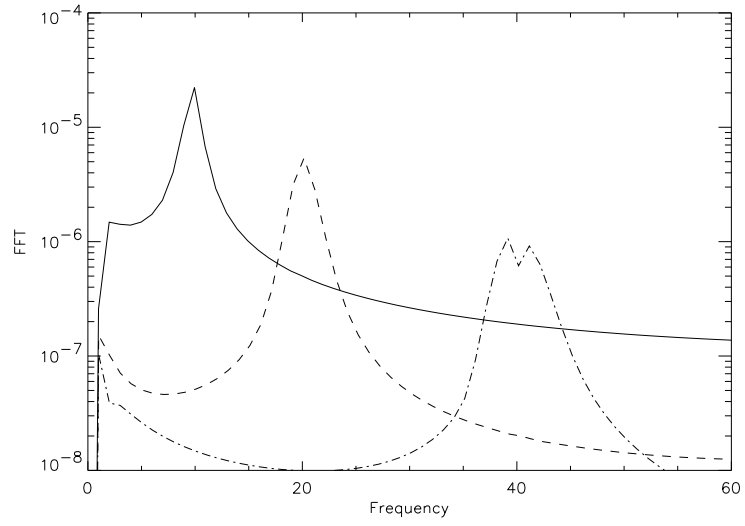

Figure 4. The Fourier transform of the data in figure 1 is shown for lattices containing of $128 \times 4 \times 4$ (——), $256 \times 4 \times 4$ (– – –), and $512 \times 4 \times 4$ (— · —) vertices. The frequency of the wave-like oscillations apparent in figure 1 are clearly proportional to N (or $1/\delta$). A quadratic drop-off in the power is also evident.