This differs from source.cdbp in that here we use the symmetry trick (by contracting with an arbitrary symmetric tensor) to set the symmetry on the connection.

```
# =============================================================================
# The connection
# =============================================================================

# --- The metric connection ---------------------------------------------------

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

g_{a b}::Metric.
g_{a}^{b}::KroneckerDelta.

\partial_{#}::PartialDerivative.

cderiv:=\partial_{c}{g_{a b}} - g_{a d}\Gamma^{d}_{b c}
                              - g_{d b}\Gamma^{d}_{a c};

Gamma:=\Gamma^{a}_{b c} -> (1/2) g^{a d} (  \partial_{b}{g_{d c}}
                                          + \partial_{c}{g_{b d}}
                                          - \partial_{d}{g_{b c}} );

@substitute!(cderiv)(@(Gamma));

@distribute!(%);
@eliminate_metric!(%);
@eliminate_kr!(%);
@canonicalise!(%);
@collect_terms!(%);
```

$$cderiv := \partial_{c}g_{ab} - g_{ad}\Gamma^{d}{}_{bc} - g_{db}\Gamma^{d}{}_{ac}$$

$$Gamma := \Gamma^{a}{}_{bc} \rightarrow \frac{1}{2}g^{ad}\left(\partial_{b}g_{dc} + \partial_{c}g_{bd} - \partial_{d}g_{bc}\right)$$

$$cderiv := \partial_{c}g_{ab} - \frac{1}{2}g_{ad}g^{de}\left(\partial_{b}g_{ec} + \partial_{c}g_{be} - \partial_{e}g_{bc}\right) - \frac{1}{2}g_{db}g^{de}\left(\partial_{a}g_{ec} + \partial_{c}g_{ae} - \partial_{e}g_{ac}\right)$$

$$cderiv := \partial_{c}g_{ab} - \frac{1}{2}g_{ad}g^{de}\partial_{b}g_{ec} - \frac{1}{2}g_{ad}g^{de}\partial_{c}g_{be} + \frac{1}{2}g_{ad}g^{de}\partial_{e}g_{bc} - \frac{1}{2}g_{db}g^{de}\partial_{a}g_{ec} - \frac{1}{2}g_{db}g^{de}\partial_{c}g_{ae} + \frac{1}{2}g_{db}g^{de}\partial_{e}g_{ac}$$

$$cderiv := \partial_{c}g_{ab} - \frac{1}{2}g_{a}{}^{e}\partial_{b}g_{ec} - \frac{1}{2}g_{a}{}^{e}\partial_{c}g_{be} + \frac{1}{2}g_{a}{}^{e}\partial_{e}g_{bc} - \frac{1}{2}g_{b}{}^{e}\partial_{a}g_{ec} - \frac{1}{2}g_{b}{}^{e}\partial_{c}g_{ae} + \frac{1}{2}g_{b}{}^{e}\partial_{e}g_{ac}$$

$$cderiv := \partial_{c}g_{ab} - \frac{1}{2}\partial_{b}g_{ac} - \frac{1}{2}\partial_{c}g_{ba} + \frac{1}{2}\partial_{a}g_{bc} - \frac{1}{2}\partial_{a}g_{bc} - \frac{1}{2}\partial_{c}g_{ab} + \frac{1}{2}\partial_{b}g_{ac}$$

$$cderiv := \partial_{c}g_{ab} - \frac{1}{2}\partial_{b}g_{ac} - \frac{1}{2}\partial_{c}g_{ab} + \frac{1}{2}\partial_{a}g_{bc} - \frac{1}{2}\partial_{a}g_{bc} - \frac{1}{2}\partial_{c}g_{ab} + \frac{1}{2}\partial_{b}g_{ac}$$

$$cderiv := 0$$

```
# ==============================================================================
# One covariant derivative
# ==============================================================================

@reset.

# --- Covariant differentiation ------------------------------------------------

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial_{#}::PartialDerivative.

# --- construct the scalar v_{a} A^{a}  ---------------------------------------

scalar:=v_{a} A^{a}:

# --- compute the derivative  -------------------------------------------------

derivD:=D^{c}\partial_{c}{@(scalar)}:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(D^{a}\partial_{a}{A^{b}} -> -\Gamma^{b}_{a c}A^{a}D^{c}):
@substitute!(%)(D^{a}\partial_{a}{D^{b}} -> -\Gamma^{b}_{a c}D^{a}D^{c}):
@prodsort!(%):
@rename_dummies!(%):
@canonicalise!(%):

# --- tidy up and display the results  ----------------------------------------

@factor_out!(scalar){A^{a}}:
@factor_out!(scalar){D^{a}};

@factor_out!(derivD){A^{a}}:
@factor_out!(derivD){D^{a}};
```

$$scalar \; := v_a A^a$$

$$derivD \; := A^a D^b \left(\partial_b v_a \; - \Gamma^c{}_{ab} v_c\right)$$

```
# ==============================================================================
# Commutation of covariant derivatives and the Riemann tensor
# ==============================================================================

@reset.

# --- covariant differentiation and the Riemann tensor ------------------------

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\partial_{#}::PartialDerivative.

# --- force Gamma to be symmetric in its lower two indices --------------------

# \Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

# --- construct the scalar v_{a} A^{a}  ---------------------------------------

scalar:=v_{a} A^{a}:

# --- compute the covariant derivative in the direction of D^a  ---------------

derivD:=D^{c}\partial_{c}{@(scalar)}:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(D^{a}\partial_{a}{A^{b}} -> -\Gamma^{b}_{a c}A^{a}D^{c}):
@prodsort!(%):
@rename_dummies!(%):
@canonicalise!(%):

# --- compute the covariant derivative in the direction of E^a  ---------------

derivDE:=E^{c}\partial_{c}{@(derivD)}:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(E^{a}\partial_{a}{A^{b}} -> -\Gamma^{b}_{a c}A^{a}E^{c}):
@substitute!(%)(E^{a}\partial_{a}{D^{b}} -> -\Gamma^{b}_{a c}D^{a}E^{c}):
@prodsort!(%):
@rename_dummies!(%):
@canonicalise!(%);

# --- copy to derivED then swap the order of the derivatives ------------------

derivED:=@(derivDE):

@substitute!(%)(E^{a} -> F^{a}):
@substitute!(%)(D^{a} -> E^{a}):
@substitute!(%)(F^{a} -> D^{a});

# --- subtract ----------------------------------------------------------------

diff:=@(derivDE) - @(derivED);

# --- trick to impose zero torsion (symmetric connection) ---------------------

G_{a b}::Symmetric.

@substitute!(%)(\Gamma^{a}_{b c} ->  G^{a} G_{b c});
@prodsort!(%):
@canonicalise!(%):
@collect_terms!(%):
@substitute!!(%)(G^{a} G_{b c} -> \Gamma^{a}_{b c});

# --- tidy up and display the results  ----------------------------------------
```

```
{A^{a},D^{a},E^{a},v_{a},\Gamma^{a}_{b c}}::SortOrder.

@prodsort!(%):
@rename_dummies!(%):

@factor_out!(%){A^{a}}:
@factor_out!(%){D^{a}}:
@factor_out!(%){E^{a}}:
@factor_out!(%){v_{a}};

@print["A^a D^b E^c (v_{a;b;c}-v_{a;c;b})="~@(diff)];
```

$$\begin{aligned} derivDE \ := & -A^{a}D^{b}E^{c}\Gamma^{d}{}_{ac}\partial_{b}v_{d} - A^{a}D^{b}E^{c}\Gamma^{d}{}_{bc}\partial_{d}v_{a} + A^{a}D^{b}E^{c}\partial_{bc}v_{a} + A^{a}D^{b}E^{c}\Gamma^{d}{}_{ac}\Gamma^{e}{}_{db}v_{e} \\ & + A^{a}D^{b}E^{c}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bc}v_{d} - A^{a}D^{b}E^{c}\partial_{c}\Gamma^{d}{}_{ab}\,v_{d} - A^{a}D^{b}E^{c}\Gamma^{d}{}_{ab}\partial_{c}v_{d} \end{aligned}$$

$$\begin{aligned} derivED \ := & -A^{a}E^{b}D^{c}\Gamma^{d}{}_{ac}\partial_{b}v_{d} - A^{a}E^{b}D^{c}\Gamma^{d}{}_{bc}\partial_{d}v_{a} + A^{a}E^{b}D^{c}\partial_{bc}v_{a} + A^{a}E^{b}D^{c}\Gamma^{d}{}_{ac}\Gamma^{e}{}_{db}v_{e} \\ & + A^{a}E^{b}D^{c}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bc}v_{d} - A^{a}E^{b}D^{c}\partial_{c}\Gamma^{d}{}_{ab}\,v_{d} - A^{a}E^{b}D^{c}\Gamma^{d}{}_{ab}\partial_{c}v_{d} \end{aligned}$$

$$\begin{aligned} diff \ := & -A^{a}D^{b}E^{c}\Gamma^{d}{}_{ac}\partial_{b}v_{d} - A^{a}D^{b}E^{c}\Gamma^{d}{}_{bc}\partial_{d}v_{a} + A^{a}D^{b}E^{c}\partial_{bc}v_{a} + A^{a}D^{b}E^{c}\Gamma^{d}{}_{ac}\Gamma^{e}{}_{db}v_{e} + A^{a}D^{b}E^{c}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bc}v_{d} \\ & - A^{a}D^{b}E^{c}\partial_{c}\Gamma^{d}{}_{ab}\,v_{d} - A^{a}D^{b}E^{c}\Gamma^{d}{}_{ab}\partial_{c}v_{d} + A^{a}E^{b}D^{c}\Gamma^{d}{}_{ac}\partial_{b}v_{d} + A^{a}E^{b}D^{c}\Gamma^{d}{}_{bc}\partial_{d}v_{a} - A^{a}E^{b}D^{c}\partial_{bc}v_{a} \\ & - A^{a}E^{b}D^{c}\Gamma^{d}{}_{ac}\Gamma^{e}{}_{db}v_{e} - A^{a}E^{b}D^{c}\Gamma^{d}{}_{ae}\Gamma^{e}{}_{bc}v_{d} + A^{a}E^{b}D^{c}\partial_{c}\Gamma^{d}{}_{ab}\,v_{d} + A^{a}E^{b}D^{c}\Gamma^{d}{}_{ab}\partial_{c}v_{d} \end{aligned}$$

$$\begin{aligned} diff \ := & -A^{a}D^{b}E^{c}G^{d}G_{ac}\partial_{b}v_{d} - A^{a}D^{b}E^{c}G^{d}G_{bc}\partial_{d}v_{a} + A^{a}D^{b}E^{c}\partial_{bc}v_{a} + A^{a}D^{b}E^{c}G^{d}G_{ac}G^{e}G_{db}v_{e} \\ & + A^{a}D^{b}E^{c}G^{d}G_{ae}G^{e}G_{bc}v_{d} - A^{a}D^{b}E^{c}\partial_{c}\left(G^{d}G_{ab}\right)v_{d} - A^{a}D^{b}E^{c}G^{d}G_{ab}\partial_{c}v_{d} \\ & + A^{a}E^{b}D^{c}G^{d}G_{ac}\partial_{b}v_{d} + A^{a}E^{b}D^{c}G^{d}G_{bc}\partial_{d}v_{a} - A^{a}E^{b}D^{c}\partial_{bc}v_{a} - A^{a}E^{b}D^{c}G^{d}G_{ac}G^{e}G_{db}v_{e} \\ & - A^{a}E^{b}D^{c}G^{d}G_{ae}G^{e}G_{bc}v_{d} + A^{a}E^{b}D^{c}\partial_{c}\left(G^{d}G_{ab}\right)v_{d} + A^{a}E^{b}D^{c}G^{d}G_{ab}\partial_{c}v_{d} \end{aligned}$$

$$diff \ := A^{a}D^{b}E^{c}\Gamma^{d}{}_{ac}\Gamma^{e}{}_{bd}v_{e} - A^{a}D^{b}E^{c}\partial_{c}\Gamma^{d}{}_{ab}\,v_{d} - A^{a}D^{b}E^{c}\Gamma^{d}{}_{ab}\Gamma^{e}{}_{cd}v_{e} + A^{a}D^{b}E^{c}\partial_{b}\Gamma^{d}{}_{ac}\,v_{d}$$

$$diff \ := A^{a}D^{b}E^{c}v_{d}\left(\Gamma^{e}{}_{ac}\Gamma^{d}{}_{be} - \partial_{c}\Gamma^{d}{}_{ab} - \Gamma^{e}{}_{ab}\Gamma^{d}{}_{ce} + \partial_{b}\Gamma^{d}{}_{ac}\right)$$

$$A^{a}D^{b}E^{c}\left(v_{a;b;c} - v_{a;c;b}\right) = A^{a}D^{b}E^{c}v_{d}\left(\Gamma^{e}{}_{ac}\Gamma^{d}{}_{be} - \partial_{c}\Gamma^{d}{}_{ab} - \Gamma^{e}{}_{ab}\Gamma^{d}{}_{ce} + \partial_{b}\Gamma^{d}{}_{ac}\right)$$

```
# ==============================================================================
# Gauss equation for 3d/4d curvatures
# ==============================================================================

@reset.

# --- The Gauss equation --------------------------------------------------

::PostDefaultRules( @@collect_terms!(%) ).
{a,b,c,d,e,f,g,i,j,k,l,m,n,o,p,q,r,s,t,u#}::Indices(position=independent).

\nabla_{#}::Derivative.

K_{a b}::Symmetric.
g^{a}_{b}::KroneckerDelta.

# --- define the projection operator -------------------------------------------

hab:=h^{a}_{b} -> g^{a}_{b} - n^{a} n_{b}:

# --- 3-covariant derivative obtained by projection on 4-covariant derivative -

vpq:=v_{p q} -> h^{a}_{p}h^{b}_{q}\nabla_{b}{v_{a}}:

# --- compute 3-curvature by commutation of covariant derivatives -------------

vpqr:=h^{a}_{p}h^{b}_{q}h^{c}_{r} ( \nabla_{c}{v_{a b}} - \nabla_{b}{v_{a c}} ):

@substitute!(vpq)(@(hab)):
@substitute!(vpqr)(@(vpq)):

@distribute!(%):
@prodrule!(%):
@distribute!(%):
@eliminate_kr!(%):

# --- standard substitutions --------------------------------------------------

@substitute!(%)(h^{a}_{b} n^{b} -> 0):
@substitute!(%)(h^{a}_{b} n_{a} -> 0):
@substitute!(%)(\nabla_{a}{g^{b}_{c}} -> 0):
@substitute!(%)(n^{a}\nabla_{b}{v_{a}} -> -v_{a}\nabla_{b}{n^{a}}):
@substitute!(%)(v_{a}\nabla_{b}{n^{a}} -> v_{p}h^{p}_{a}\nabla_{b}{n^{a}}):
@substitute!(%)(h^{p}_{a}h^{q}_{b}\nabla_{p}{n_{q}} -> K_{a b}):
@substitute!(%)(h^{p}_{a}h^{q}_{b}\nabla_{p}{n^{b}} -> K_{a}^{q}):

# --- tidy up and display the results -----------------------------------------

{h^{a}_{b},\nabla_{a}{v_{b}}}::SortOrder.

@prodsort!(%):
@rename_dummies!(%):
@canonicalise!(%):
@factor_out!!(%){h^{a}_{b}};
```

$$vpqr := h^{a}{}_{p} h^{b}{}_{q} h^{c}{}_{r} \left(\nabla_{c}\nabla_{b} v_{a} - \nabla_{b}\nabla_{c} v_{a}\right) + K_{pr} K_{q}{}^{a} v_{a} - K_{pq} K_{r}{}^{a} v_{a}$$