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Fast algorithms for computing defects and their derivatives in the Regge calculus.

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Abstract

Any practical attempt to solve the Regge equations, these being a large system of non-linear algebraic equations, will almost certainly employ a Newton-Raphson like scheme. In such cases it is essential that efficient algorithms be used when computing the defect angles and their derivatives with respect to the leg-lengths. The purpose of this paper is to present details of such an algorithm.

1 The Regge calculus

In its pure form (there are variations) the Regge calculus is a theory of gravity in which the spacetime is built from a (possibly infinite) collection of non-overlapping flat 4-simplices (plus further low level constraints that ensure that the manifold remains 4-dimensional everywhere). This construction provides a clear distinction between the topological and metric properties of such spacetimes. The topology is encoded in the way the 4-simplices are tied together while the metric is expressed as an assignment of lengths to each leg of the simplicial lattice.

The Regge action on a simplicial lattice is defined by

$$I_R = \sum_{\sigma_2} 2\left(\theta A\right)_{\sigma_2} \tag{1.1}$$

where θ_{σ_2} is the defect on a typical triangle σ_2, A_{σ_2} is the area of the triangle and the sum includes every triangle in the simplicial lattice. This action is a function of the leg lengths $L^2(\sigma_1)$ and its extremization with respect to a typical leg length leads to

$$0 = \sum_{\sigma_2(\sigma_1)} \left(\theta \frac{\partial A}{\partial L^2} \right)_{\sigma_2} \tag{1.2}$$

where the sum includes each triangle σ_2 that contains the leg σ_1. This set of equations, one per leg, are known as the Regge field equations for the lattice. Constructing a Regge simplicial lattice will require frequent computations of the defects as well as frequent solutions of the above Regge equations, a large system of non-linear algebraic equations. These are both nontrivial tasks.

Most articles on the Regge Calculus are concerned with formal issues, such as the nature of its convergence to the continuum [1, 2, 3, 4, 5], its use as a tool in quantum gravity [6, 7], the relationship of the Regge equations to the ADM equations [8, 9] and so on (for further reading see [10, 11]).

When people turn to numerical computations in the Regge calculus they usually employ a sufficiently simple lattice that the issue of computational efficiency is not a major concern. But if the Regge calculus is to ever prove useful in the study of discrete gravity, in either classical or quantum contexts, then large scale simulations will be required and consequently the issue of computational efficiency will be paramount.

To the best of the author's knowledge, there are no papers that pose the specific question *What is the best way to compute the defects?*. By *best* we mean minimal computational effort, that the cpu time required to compute each defect be as short as possible. The main purpose of this article is to address that question. We do not claim that the algorithms presented here are optimal but rather that they are a significant improvement on contemporary methods. In particular we will present details of an algorithm for computing the derivatives of the defects at virtually no extra cost above that required to compute the defects.

This paper is organised as follows. In section (2) we introduce various elements such as a coordinate frame (the standard frame) and various vectors (e.g., normal vectors) required to compute the defects. Explicit equations for the defects are given in section (3), and their derivatives in section (4). Finally, in section (5), we present some simple timings for our algorithms against centred finite differences and an algorithm by Dittrich et al. [12].

We will employ Greek characters for coordinates indices and Roman characters as labels for the vertices of the lattice.

2 The standard frame

A simple coordinate system for a typical 4-simplex such as (01234) can be constructed as follows. We start by adopting the four edge vectors at (0) as a basis for the tangent space at (0). Let $(0x)$ be the vector that connects (0) to a typical point (x) within the 4-simplex. Since the metric of the 4-simplex is flat we can uniquely project $(0x)$ onto the basis, that is if we take e_i to be the basis vector based on the edge $(0i)$, then we can write

$$(0x) = x^1 e_1 + x^2 e_2 + x^3 e_3 + x^4 e_4 \tag{2.1}$$

for some set of numbers x^i. We take the $(x^\mu) = (x^1, x^2, x^3, x^4)$ to be the *standard coordinates* of the point x. Note that these coordinates are similar to but distinct from another popular choice known as barycentric coordinates (see [9, 13, 14]).

The coordinates just described are not new and have been used by others in the field (see for example [8, 15]).

Let x_i^μ , $i = 0, 1, 2, 3, 4$ be the standard coordinates for the five vertices of the the 4-simplex (01234). Then from the above we easily see that $x_0^\mu = 0$ while $x_i^\mu = \delta_i^\mu$, $i = 1, 2, 3, 4$. Finally, any point within the 4-simplex will have coordinates x^μ subject to the constraints $0 \leq x^\mu \leq 1$ and $\sum_\mu x^\mu \leq 1$. Note that $\sum_\mu x^\mu = 1$ describes the tangent plane to the face opposite the vertex (0).

The flat metric for this 4-simplex can be constructed from the leg-lengths by choosing a constant symmetric 4×4 matrix $g_{\mu\nu}$ such that

$$L_{ij}^2 = g_{\mu\nu} \Delta x_{ij}^\mu \Delta x_{ij}^\nu \qquad\qquad i, j = 0, 1, 2, 3, 4$$

where $\Delta x_{ij}^\mu := x_j^\mu - x_i^\mu$. This leads to a 10×10 system of equations for the $g_{\mu\nu}$ and the solution is easily seen to be

$$g_{ij} = \frac{1}{2} \left(L_{0i}^2 + L_{0j}^2 - L_{ij}^2 \right) \qquad i, j = 1, 2, 3, 4 \tag{2.2}$$

provided we take $L_{ij} = 0$ when $i = j$. (Note also the slight abuse of notation used here, the left hand side should use Greek letters.)

There are of course many other standard frames that could be constructed within one 4-simplex (for example, by choosing a vertex other than (0) as the origin). Clearly any such frame can be constructed from that given above by a simple permutation of the labels assigned to the vertices. Thus if g and g' are the metrics in two such standard frames then their components will be related by a transformation of the form $g'_{\alpha\beta} = M^{\mu}{}_{\alpha} M^{\nu}{}_{\beta} g_{\mu\nu}$ for some simple matrix $M^{\alpha}{}_{\beta}$. Likewise, the components of a vector will transform according to $v^{\mu} = M^{\mu}{}_{\nu} v'^{\nu}$. The matrix M can be easily found by applying this last equation to the basis vectors of one of the standard frames.

2.1 Volumes

The 4-volume V_4 of a typical 4-simplex is given by

$$V_4 = \iiiint\limits_{\substack{0<x^1,x^2,x^3,x^4<1 \\ 0<x^1+x^2+x^3+x^4<1}} \sqrt{|g|}\, dx^1 dx^2 dx^3 dx^4 = \frac{1}{4!}\sqrt{|g|} \tag{2.3}$$

where $g = \det(g_{\mu\nu})$ in the standard frame. Similar expressions apply for any $n-$simplex. If we denote the measure of an $n-$simplex by V_n then

$$V_n = \frac{1}{n!}\sqrt{|g|} \tag{2.4}$$

where $g = \det(g_{\mu\nu})$ in the standard frame for the $n-$simplex.

2.2 Normal vectors

It is not hard to verify that the unit vectors n_3 and m_3 of figure (1) have the following components (in the standard frame based on (01234))

$$n_3^{\mu} = -\operatorname{sign}\left(g^{44}\right) \frac{g^{4\mu}}{\sqrt{|g^{44}|}} \tag{2.5}$$

$$m_3^{\mu} = \operatorname{sign}\left(h^{33}\right) \frac{h^{3\mu}}{\sqrt{|h^{33}|}} \quad \text{where} \quad h^{\mu\nu} = g^{\mu\nu} - \frac{g^{4\mu} g^{4\nu}}{g^{44}} \tag{2.6}$$

The corresponding n and m vectors for each of the remaining faces can be constructed using simple coordinate transformations as suggested earlier in section 2.

2.3 Dihedral angles

A popular way to compute the dihedral angle ϕ_{34} is to use a ratio of volumes,

$$\sin\phi_{34} = \frac{4}{3}\frac{V_4 V_2}{V_{3,1}V_{3,2}} \tag{2.7}$$

where $V_{3,1}$ and $V_{3,2}$ are the volumes of the two tetrahedral faces of σ_4 while V_4 is the 4-volume of σ_4 and V_2 is the area of σ_2. Using this equation in a Lorentzian simplicial lattice does require some care when choosing the correct complex valued branch of the inverse sine function (see Wheeler [16] and Sorkin [17] for full details).

We shall take a different (but equivalent) approach to Wheeler and Sorkin by working entirely with real valued expressions. See [18, 15] for details of computing angles within a Lorentzian simplicial lattice. By forming suitable dot-products of the n and m vectors we find that the dihedral angle can be computed as follows. For a *timelike* bone,

$$\phi_{34} = \arccos\left(m_3^\mu m_{4\mu}\right) \tag{2.8}$$

while for a *spacelike* bone,

$$\phi_{34} = \operatorname{sign}\left(m_3^\mu m_{3\mu}\right)\operatorname{arcsinh}\rho_{12} \tag{2.9}$$

where ρ_{12} is computed from

$$\rho_{12} = \begin{cases} \operatorname{sign}\left(n_3^\nu m_{2\nu}\right) m_3^\mu m_{4\mu} & \text{when } |m_3^\mu m_{4\mu}| < |n_3^\mu m_{4\mu}| \\ \operatorname{sign}\left(m_3^\nu m_{2\nu}\right) n_3^\mu m_{4\mu} & \text{in all other cases} \end{cases} \tag{2.10}$$

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In the previous equation $m_{2\nu}$ should be replaced with $m_{4\nu}$. Here is the corrected equation.

$$\rho_{12} = \begin{cases} \operatorname{sign}\left(n_3^\nu m_{4\nu}\right) m_3^\mu m_{4\mu} & \text{when } |m_3^\mu m_{4\mu}| < |n_3^\mu m_{4\mu}| \\ \operatorname{sign}\left(m_3^\nu m_{4\nu}\right) n_3^\mu m_{4\mu} & \text{in all other cases} \end{cases} \tag{2.10}$$

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3 Defect angles

The defect θ_{σ_2} on a typical bone σ_2 is defined by

$$\theta_{\sigma_2} = \begin{cases} 2\pi - \sum\limits_{\sigma_4(\sigma_2)} (\phi_{34})_{\sigma_4} & \text{for } \sigma_2 \text{ timelike} \\ \quad - \sum\limits_{\sigma_4(\sigma_2)} (\phi_{34})_{\sigma_4} & \text{for } \sigma_2 \text{ spacelike} \end{cases} \tag{3.1}$$

where the sum includes all of the 4-simplices σ_4 attached to the bone σ_2.

4 Derivatives of defects

Choose one 4-simplex $\sigma_4 := (01234)$ and consider the angle ϕ_{34} between the pair of 3-simplices $\sigma_3(4) := (0123)$ and $\sigma_3(3) := (0124)$. Suppose for the moment that $\sigma_2 := (012)$ is timelike then with the orientations for n^μ and m^μ chosen as per figure (1) we have

$$m_{4\mu} = -n_{3\mu} \sin\phi_{34} + m_{3\mu} \cos\phi_{34}$$
$$n_{4\mu} = -n_{3\mu} \cos\phi_{34} - m_{3\mu} \sin\phi_{34}$$

If some small changes are now made to the lattice (e.g., by small changes in the leg-lengths) then we must have

$$\delta m_{4\mu} = \quad n_{4\mu}\delta\phi_{34} - \delta n_{3\mu} \sin\phi_{34} + \delta m_{3\mu} \cos\phi_{34}$$
$$\delta n_{4\mu} = -m_{4\mu}\delta\phi_{34} - \delta n_{3\mu} \cos\phi_{34} - \delta m_{3\mu} \sin\phi_{34}$$

and thus

$$2\delta\phi_{34} = n_4^\mu \delta m_{4\mu} - m_4^\mu \delta n_{4\mu} + m_3^\mu \delta n_{3\mu} + n_3^\mu \delta m_{3\mu}$$

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The sign on $m_3^\mu \delta n_{3\mu}$ is incorrect. The corrected equation is

$$2\delta\phi_{34} = n_4^\mu \delta m_{4\mu} - m_4^\mu \delta n_{4\mu} - m_3^\mu \delta n_{3\mu} + n_3^\mu \delta m_{3\mu}$$

This result applies in all frames but when restricted to the standard frame it takes on a particularly simple form. In the standard frame we have

$$\begin{aligned} m_{3\mu} &= m_3\left(\delta^3_\mu + \alpha_3\delta^4_\mu\right) & \qquad n_{3\mu} &= n_3\delta^4_\mu \\ m_{4\mu} &= m_4\left(\delta^4_\mu + \alpha_4\delta^3_\mu\right) & \qquad n_{4\mu} &= n_4\delta^3_\mu \end{aligned}$$

where the m_i and n_i are normalisation factors while α_i is chosen so that $0 = n_{i\mu}m^\mu_i$. Thus we have

$$0 = m^\mu_i \delta n_{i\nu} \qquad \text{for } i = 3, 4$$

while from $0 = n_{i\mu}m^\mu_i$, $0 = n^\mu_i m_{i\mu}$ and $0 = g_{\mu\nu}n^\mu_i m^\mu_i$ we also have

$$n^\mu_i \delta m_{i\mu} = \delta g_{\mu\nu} n^\mu_i m^\mu_i \qquad \text{for } i = 3, 4$$

which leads to the following simple equation for the variations in the angle

$$2\delta\phi_{34} = \left(m^\mu_3 n^\nu_3 + m^\mu_4 n^\nu_4\right)\delta g_{\mu\nu}$$

An almost identical analysis can be applied in the case of a spacelike bone. It begins with

$$\begin{aligned} m^\mu_4 &= -n^\mu_3 \sinh\phi_{34} + m^\mu_3 \cosh\phi_{34} \\ n^\mu_4 &= -n^\mu_3 \cosh\phi_{34} + m^\mu_3 \sinh\phi_{34} \end{aligned}$$

and leads to

$$2\delta\phi_{34} = -\left(m^\mu_3 n^\nu_3 + m^\mu_4 n^\nu_4\right)\delta g_{\mu\nu}$$

To compute the derivative of the defect θ on σ_2 we need only combine the above results with the equation (3.1) for the defect to obtain[1]

$$\left(\frac{\partial\theta}{\partial L^2}\right)_{\sigma_2} = \frac{1}{2}\epsilon(\sigma_2)\sum_{\sigma_4(\sigma_2)}\left(\left(m^\mu_3 n^\nu_3 + m^\mu_4 n^\nu_4\right)\frac{\partial g_{\mu\nu}}{\partial L^2}\right)_{\sigma_4} \tag{4.1}$$

where the sum includes each of the 4-simplices attached to σ_2, L^2 is a typical (squared) leg-length and $\epsilon(\sigma_2) = -1$ for a timelike bone and $+1$ for a spacelike bone.

The utility of this equation should not be overlooked. The partial derivatives of the metric in the standard frame are simple numbers such as $1, \pm 1/2$ and

[1] A similar result, though for the case of a weak Euclidean metric, has been obtained independently by Snorre H. Christiansen, see Proposition 3.1 in [19]

zero and thus there is no extra cost (of any importance) in computing the Jacobian of the Regge equations in situ with the equations themselves. All of the terms in this equation are already in use during the computation of the defects. This is a significant advantage when it comes to solving the Regge equations by standard numerical methods, having the Jacobian at hand at no cost is a great bonus. Of course the Jacobian could be computed by finite differences but that would be extremely expensive. Each leg would need to be varied in turn and the corresponding changes in the defects recorded. If we suppose that, on average, each defect depends on N legs, then this finite-difference process will increase the computational cost of the defects by a factor of N, a significant expense (a bare minimum for N is 10, the 10 legs of one 4-simplex).

Dittrich et al. [12] have also developed an algorithm for computing the derivatives of the defects with respect to the leg lengths. Their approach is based on barycentric coordinates. The details can found in their Appendix A (see in particular equations A4,6,7,11,14 and A15).

5 Timing

To assess the gains that equation (4.1) affords over a standard finite difference algorithm we ran some simple tests using leg lengths assigned from well known metrics (e.g., Schwarzschild, plane waves, Kasner etc.). The structure of the lattice and the computation of the leg lengths are based on the methods used in an earlier paper (see [4] for full details). The results presented here are a measure of the arithmetic complexity of each algorithm. Thus it is reasonable to expect that the our results are largely independent of the details of the lattice (e.g., the size, number and location of each 4-simplex in the parent spacetime).

In our first test we compared the cost of computing the defects alone (3.1) versus computing the defects along with their derivatives (3.1,4.1). We found that computing the defects and their derivatives took approximately 11% more cpu time than computing the defects on their own. This justifies the assertion that the method proposed here does not impose a significant computational cost above that required for the defects.

In our second test we compared the computational cost of our method against a finite difference algorithm. We found that the finite difference algorithm to be vastly slower than our method, the ratio of cpu times varying linearly with the number of leg-lengths on which the defect depends. That is, the finite difference calculation would take approximately N times as long as our method to compute all of the derivatives for one defect that depends on N legs. If centred finite differences are used then the scaling would be of order $2N$ over our method. As a typical defect might depend on 50 or more legs this would be a 100 fold increase in cpu time.

Our final test compared the performance of our method against that proposed by Dittrich et al. [12]. In this test we used just one 4-simplex and computed not the derivatives of the defects but rather the derivatives of the dihedral angles. This small change was forced upon us by the nature of the Dittrich et al. algorithm – it is best suited to a computation that sweeps across each 4-simplex in the lattice while maintaining a running tally of the defects and their derivatives. We found that our algorithm was approximately twice as fast as the Dittrich et al. algorithm.

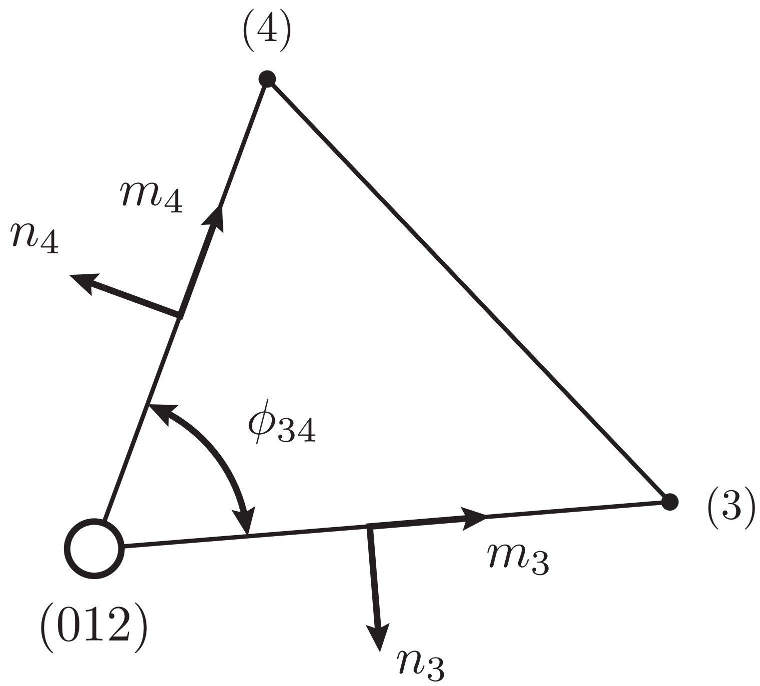

Figure 1: A typical 4-simplex (01234). Here we show the outward pointing unit-vectors n_3, n_4 and the corresponding unit tangent vectors m_3 and m_4. These vectors are used to compute the dihedral angle ϕ_{34}. The small circle is our way of noting that (012) is a 2-simplex (whereas vertices are drawn as a solid dot).

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