# Exterior differentiation in the Regge calculus

Leo Brewin
*Department of Mathematics, Monash University, Clayton, Victoria, Australia, 3168*

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Regge manifolds are piecewise continuous manifolds constructed from a finite number of basic building blocks. On such manifolds piecewise continuous forms can be defined in a way similar to differential forms on a differentiable manifold. Regge manifolds are used extensively in the construction of space-times in numerical general relativity. In this paper a definition of exterior differentiation suitable for use on piecewise continuous forms on a Regge manifold is presented. It is shown that this definition leads to a version of Stokes' theorem and also to the usual result that $d^2 = 0$. This is preceded by a discussion of certain geometrical properties of the Regge manifolds. It is shown that the version of Stokes' theorem presented here coincides with the usual definition when the Regge manifold is refined, by increasing the number of cells while keeping the total volume constant, to a smooth manifold.

## I. INTRODUCTION

It is assumed that solutions of Regge's field equations[1–5], Regge space-times, are approximations, to a degree, of an Einstein space-time, this being a differentiable solution of Einstein's field equations. This assumption is based on two facts. First, the Regge and Einstein manifolds are equivalent under a homeomorphism. Second, both sets of field equations are derived from the same action principle. It is therefore not unreasonable to expect that there should exist a correspondence between certain properties of the Regge and Einstein space-times. In particular the operation of exterior differentiation on an Einstein manifold should lead to a related operation on a Regge manifold.

The main result to be presented here is an operation on forms built on Regge manifolds which mimics the usual operation of exterior differentiation. This result[6] is presented in Sec. IV. In Secs. II and III the basic notation and formulas are presented. Finally, in Sec. V, it is shown that this definition reduces to the version usually employed on smooth manifolds.

## II. SIMPLEXES AND COMPLEXES

The fundamental building blocks for the manifolds to be considered here are known as $n$-simplexes. They may be defined in a recursive fashion as follows.

(i) A 0-simplex is a single point. This object is also called a vertex.

(ii) An $(n+1)$-simplex is constructed from a $n$-simplex by first introducing one new vertex and then joining this vertex to each of the $(n+1)$ vertices of the $n$-simplex, and second by demanding that any set of $m$ vertices $(1 < m < n+2)$ of the $(n+1)$-simplex is an $(m-1)$-simplex.

(iii) A Lorentzian $n$-simplex is obtained by imposing a flat Lorentzian metric throughout the $n$-simplex. Only those $n$-simplexes in which the induced metric on each of its $m$-simplexes $(0 < m < n)$ is also flat will be considered. This has the effect of disallowing any $n$-simplexes with curved boundaries.

The obvious method of constructing an $n$-dimensional manifold is to glue together a collection of $n$-simplexes. The resulting object is referred to as an $n$-complex. To avoid certain pathological cases the following restrictions are imposed: (i) the region of the $n$-complex common to two or more adjacent $n$-simplexes is an $m$-simplex with $0 < m < n$, and (ii) any $m$-simplex of the complex is contained within at least one $n$-simplex of the $n$-complex.

The following notation is drawn, primarily, from Seirfert and Threlfall.[7]

A typical $n$-simplex is denoted by $\sigma_n(i)$, with the index $i$ being the label which distinguishes this simplex from all other $n$-simplexes. The set of all $n$-simplexes is represented by $S_n$. Each $n$-simplex contains exactly $(n+1)$ vertices and is represented as follows:

$$\sigma_n(i) = (i_0 i_1 i_2 \cdots i_n), \quad n = 0,1,2,\dots, \tag{2.1}$$

where each of the $i_j$ is unique and is the label of a vertex of $\sigma_n(i)$. The order in which the vertices are listed is unimportant unless the simplex is oriented. One of the two possible orientations to the simplex is defined by reading the vertices in (2.1) from left to right. The opposite sense of orientation is obtained if any two vertices in the sequence are swapped. This is indicated by writing

$$(i_0 \cdots i_j \cdots i_k \cdots i_n) = -(i_0 \cdots i_k \cdots i_j \cdots i_n), \tag{2.2}$$

provided that $j \neq k$ and $j, k = 0, 1, 2, \dots, n$.

An $n$-complex is denoted by $\rho_n$ and represented by the formal sum

$$\rho_n = \sum_i a_i \sigma_n(i), \tag{2.3}$$

with each $a_i = 0, -1$, or $+1$. The coefficients $a_i$ represent whether the associated $n$-simplex is present or not and what orientation it possesses in the complex. Complexes in which certain $n$-simplexes are absent are referred to as sub- or secondary complexes. The original complex, when required, is referred to as the primary complex.



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One particularly important subcomplex is the boundary of the primary complex. It is defined as follows. First suppose that $\sigma_n(i)$ is represented as in (2.1). Then define the operation

$$\sigma_n(i/i_j) = (i_0 i_1 \cdots i_n / i_j) = \begin{cases} (-1)^j (i_0 i_1 \cdots \hat{i}_j \cdots i_n), \\ 0, \quad \text{if } i_j \text{ is not in } i_0, i_1, \ldots, i_n. \end{cases} \tag{2.4}$$

The symbol $\hat{}$ over $i_j$ indicates that the vertex $i_j$ is excluded from the list. The boundary of an $n$-simplex $\sigma_n(i)$ is defined as

$$\partial \sigma_n(i) = \sum_{i_j} \sigma_n(i/i_j). \tag{2.5}$$

Similarly for an $n$-complex

$$\begin{aligned} \rho_n &= \sum_i a_i \sigma_n(i), \\ \partial \rho_n &= \sum_{i,j} a_i \sigma_n(i/j). \end{aligned} \tag{2.6}$$

Obviously $\sigma_n(i/j)$ is also a simplex and thus the operation (2.5) may be applied twice. However from (2.2) and (2.4) it is clear that

$$\sigma_n(i/j/k) = -\sigma_n(i/k/j). \tag{2.7}$$

This leads to the usual result[7]

$$\partial^2 \sigma_n(i) = 0. \tag{2.8}$$

## III. THE METRIC FRAME OF A SIMPLEX

### A. The natural frame

One of the easiest ways of ensuring that the metric of a simplex is flat is to demand that all of the metric coefficients are constant throughout the simplex. Of course there are other frames in which the coefficients are not constant and yet the metric is flat. For simplicity such frames will be ignored.

It will be convenient to distinguish between the terms "coordinate frame" and "metric frame." The term "coordinate frame" will be used to refer to a frame possessing coordinates but not a metric. In a "metric frame" there are both coordinates and a metric. A very useful metric frame, the natural frame, will now be described.

Choose one $n$-simplex, $\sigma_n(1)$, and label its vertices from 0 to $n$. Adopt the vertex (0) as the origin of the coordinate frame. The basis vectors for this $n$-simplex are chosen as the $n$ vectors that join (0) to the remaining vertices, i.e., along those legs connected to the origin. The coordinate frame is chosen such that the coordinates of the vertices are

$$\begin{aligned} \sigma_0(0) &= (0) \quad &:& \quad (v_0^\mu) = (0,0, \ldots ,0), \\ \sigma_0(1) &= (1) \quad &:& \quad (v_1^\mu) = (1,0, \ldots ,0), \\ \sigma_0(2) &= (2) \quad &:& \quad (v_2^\mu) = (0,1, \ldots ,0), \\ & &\vdots& \\ \sigma_0(n) &= (n) \quad &:& \quad (v_n^\mu) = (0,0, \ldots ,1). \end{aligned}$$

In short

$$v_n^\mu = \begin{cases} 0, & \text{if } n = 0, \quad \mu = 1,2, \ldots ,n, \\ \delta_n^\mu, & \text{if } n > 0, \quad \mu = 1,2, \ldots ,n. \end{cases} \tag{3.1}$$

Denote the basis vectors by $\mathbf{e}_\mu$. Then any point $P$ in the simplex is described by the vector

$$\mathbf{P} = x^\mu \mathbf{e}_\mu, \quad \text{with each } x^\mu \geqslant 0 \text{ and } \sum_\mu x^\mu \leqslant 1. \tag{3.2}$$

The requirement that $\Sigma_\mu x^\mu \leqslant 1$ ensures that the vector does not pass through the face opposite the origin. This completes the construction of a coordinate frame for this simplex. A metric frame will now be constructed by introducing the leg lengths $L_{ij}$ and the metric components $g_{\mu\nu}$.

Denote the proper distance between the vertices $(i)$ and $(j)$ by $L_{ij}$. If all the $g_{\mu\nu}$ are known then, using (3.1), the $L_{ij}$ would be computed as

$$\begin{aligned} L_{0i}^2 &= g_{ii}, & 1 \leqslant i \leqslant n, \\ L_{ij}^2 &= (g_{ii} - 2g_{ij} + g_{jj}), & 1 \leqslant i,j \leqslant n \text{ and } i \neq j. \end{aligned} \tag{3.3}$$

In this instance there is no summation over repeated indices. Solve these equations for the $g_{\mu\nu}$ to obtain

$$\begin{aligned} g_{ii} &= L_{01}^2, & 1 \leqslant i \leqslant n, \\ g_{ij} &= (L_{0i}^2 + L_{0j}^2 - L_{ij}^2)/2, & 1 \leqslant i,j \leqslant n \text{ and } i \neq j. \end{aligned} \tag{3.4}$$

So far nothing has been said about the signature of the metric. For a physically realistic space-time, in the sense of general relativity, the signature must be Lorentzian (i.e., $- + + +$). Thus it is clear that not all of the leg lengths can be specified without restriction. However, this restriction is somewhat weak for it is possible, in all but a few exceptions, to make small arbitrary changes in the $L_{ij}$ and yet not change the signature. Therefore assume that the signature is Lorentzian for each and every $n$-simplex of the primary complex. In a later paper a technique of constructing an $n$-simplex will be described by which the signature may be guaranteed to be Lorentzian.

A knowledge of the $g_{\mu\nu}$ also enables the computation of areas and volumes of $n$-simplexes. Define the measure of an $n$-simplex, $\sigma_n$, as the $n$-fold integral

$$M(\sigma_n) = \int_{n\text{-simplex}} ({}^n g)^{1/2} \, d^n x, \tag{3.5}$$

where ${}^n g$ is the determinant of the $g_{\mu\nu}$. The limits of integration are easily deduced from (3.1). For example, for a four-simplex, the four-dimensional measure is

$$M(\sigma_4) = ({}^4 g)^{1/2} \times \int_0^1 \int_0^{1-x^1} \int_0^{1-x^1-x^2} \int_0^{1-x^1-x^2-x^3} dx^4\, dx^3\, dx^2\, dx^1,$$

since ${}^4 g$ is a constant. In this instance the repeated integral has the value $\frac{1}{4}!$, in the general case of (3.5) the value is $1/n!$. Since the signature of the metric may be indefinite it is possible to obtain an imaginary value for $M(\sigma_n)$. This is an unnecessary complication and will be avoided by using the absolute value of ${}^n g$ in (3.5). Thus the measure of an $n$-simplex, $\sigma_n$, is

$$M(\sigma_n) = (1/n!)\,(\mathrm{abs}({}^n g))^{1/2}. \tag{3.6}$$

### B. The general frame

On occasions it may be useful to employ a frame other than the natural frame. For example, if a study of the proper-



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ties of a group of simplexes is to be made then it may be necessary to build a metric frame covering all simplexes of the group. Clearly the natural frame is inappropriate in this example. The construction of a general class of metric frames will now be discussed. Once again assume that the $g_{\mu\nu}$ are constant throughout each $n$-simplex and that the basis vectors are chosen as $\partial/\partial x^{\mu}$.

Consider a complex with one or more $n$-simplexes. To each vertex, $\sigma_0(i)$, of this complex, associate $n$ coordinates, $x^{\mu}(i)$. Provided the topology of this complex is not too peculiar, it should be possible to choose these coordinates so that, for each and every $n$-simplex, the coordinates constitute a coordinate frame for that simplex. This condition simply ensures that locally (i.e., within one $n$-simplex) the coordinate frame is $n$-dimensional. Assume that such a choice can and has been made.

The components of the vector joining $\sigma_0(j)$ to $\sigma_0(i)$, denoted by $L^{\mu}(ij)$, have the values

$$L^{\mu}(ij) = x^{\mu}(i) - x^{\mu}(j). \tag{3.7}$$

The values of the $g_{\mu\nu}$ are obtained by solving the equation

$$L^2_{ij} = g_{\mu\nu} L^{\mu}(ij) L^{\nu}(ij). \tag{3.8}$$

Since there are $n(n+1)/2$ leg lengths in each $n$-simplex and a similar number of $g_{\mu\nu}$'s there may exist a unique solution of (3.8). That a unique solution does exist is guaranteed by the earlier requirement that the coordinate frame be everywhere $n$-dimensional.

Notice that the values of the $g_{\mu\nu}$ need not be the same in each $n$-simplex. Thus there may be discontinuous changes in the $g_{\mu\nu}$ across the interfaces between pairs of $n$-simplexes. Consequently there results a possible ambiguity in the process of raising and lowering indices. For example suppose the leg $(ij)$ is common to two $n$-simplexes. Then the values of $L^{\mu}(ij)$ with the index lowered may depend upon the choice of simplex in which the computation was performed. There is of course no ambiguity in the $L^{\mu}(ij)$. It would therefore be inaccurate to write $L_{\mu}(ij)$ as the lowered version of $L^{\mu}(ij)$, however, in most applications it will be clear which $n$-simplex is intended.

Consider one $m$-simplex $\sigma_m = (i_0 i_1 \dots i_m)$ in an $(m+1)$-complex and now define the following quantities

$$L^{\mu}_j = L^{\mu}(i_0 i_j), \quad \text{for } j = 1,2,\dots,m,$$

$$L^{\mu_1\mu_2\cdots\mu_m}(\sigma_m) = e^{\nu_1\nu_2\cdots\nu_m}_{1\,2\,\cdots\,m} L^{\mu_1}_{\nu_1} L^{\mu_2}_{\nu_2} \cdots L^{\mu_m}_{\nu_m}, \tag{3.9}$$

with $e^{\mu\nu\cdots\rho}_{ij\cdots k} = +1, -1, 0$ when $(\mu\nu\cdots\rho)$ is either an even, odd, or a nonpermutation of $(ij\cdots k)$, respectively. For example, for $m=3$,

$$L^{\mu\nu\rho}(\sigma_3) = L^{\mu}_1 L^{\nu}_2 L^{\rho}_3 - L^{\nu}_1 L^{\mu}_2 L^{\rho}_3 + L^{\nu}_1 L^{\rho}_2 L^{\mu}_3$$
$$- L^{\rho}_1 L^{\nu}_2 L^{\mu}_3 + L^{\rho}_1 L^{\mu}_2 L^{\nu}_3 - L^{\mu}_1 L^{\rho}_2 L^{\nu}_3.$$

The following expressions, shown only for $m=3$ but easily generalized, are all derived from the definition (3.9):

$$L^{\mu\nu\rho}(\sigma_3) = e^{ijk}_{123} L^{\mu}_i L^{\nu}_j L^{\rho}_k, \tag{3.10a}$$

$$L^{\mu\nu\rho}(\sigma_3) = e^{ijk}_{123} N^{\mu}_i L^{\nu}_j L^{\rho}_k, \tag{3.10b}$$

$$L^{\mu\nu\rho}(\sigma_3) = e^{\mu\nu\rho}_{\alpha\beta\gamma} L^{\alpha}(\sigma_1) L^{\beta\gamma}(\sigma_2), \tag{3.10c}$$

$$L^{\mu\nu\rho}(\sigma_3) = e^{\mu\nu\rho}_{\alpha\beta\gamma} N^{\alpha}(\sigma_1) L^{\beta\gamma}(\sigma_2), \tag{3.10d}$$

with $\sigma_2$ = one face of $\sigma_3$, $\sigma_1$ = one leg of $\sigma_3$ but not of $\sigma_2$, $N^{\alpha}(\sigma_2)$ = the projection of $L^{\alpha}(\sigma_1)$ onto the normal to $\sigma_2$, and $N^{\alpha}_i$ = the projection of $L^{\alpha}_i$ onto the normal of its adjacent face.

Now let $L(\sigma_n)$ = the measure of the parallel $n$-cube formed from the $n$ legs

$(i_0 i_1),\dots,(i_0 i_n)$ of $\sigma_n = (i_0 i_1 \cdots i_n)$.

It is well known that

$$n! L^2(\sigma_n) = L_{(\mu)}(\sigma_n) L^{(\mu)}(\sigma_n), \quad \text{with } (\mu) = (\mu_1 \mu_2 \cdots \mu_n). \tag{3.11}$$

Alternatively $L(\sigma_n)$ can be computed by an integration like that in (3.5). In this case the limits of integration must now be chosen to cover an $n$-cube rather than an $n$-simplex. The result of this integration is similar to (3.6) with the exclusion of the $n!$, thus

$$M(\sigma_n) = (1/n!) L(\sigma_n). \tag{3.12}$$

As the measure of any simplex must be a property of that simplex alone, it follows that any ambiguity in the computation of, for example, $L_{\mu\nu}(\sigma_2)$ must be resolved in the process of computing $L(\sigma_2)$. This circumstance is also evident from the fact that (3.11) is a scalar equation.

For the remaining part of this section it is assumed that the dimension of the complex is 3. After presenting and justifying the definition of exterior differentiation the result will be extended to higher dimensions.

Consider a typical three-simplex $\sigma_3$. Suppose that $\sigma_3$ has $\sigma_2$ as a base and that $\sigma_1$ is a leg of $\sigma_3$ but not of $\sigma_2$. Then from (3.10c)

$$L^{\rho\mu\nu}(\sigma_3) = L^{\rho}(\sigma_1) L^{\mu\nu}(\sigma_2)$$
$$- L^{\mu}(\sigma_1) L^{\rho\nu}(\sigma_2) + L^{\nu}(\sigma_1) L^{\rho\mu}(\sigma_2). \tag{3.13}$$

Now suppose that $n_{\rho}(\sigma_2)$ is a unit vector normal to the base $\sigma_2$. Then a contraction of (3.13) with $n_{\rho}$ results in

$$n_{\rho}(\sigma_2) L^{\rho\mu\nu}(\sigma_3) = n_{\rho}(\sigma_2) L^{\rho}(\sigma_1) L^{\mu\nu}(\sigma_2).$$

However, $n_{\rho}(\sigma_2) L^{\rho}(\sigma_1)$ is the projection of $L^{\rho}$ in the direction of $n_{\rho}$, which is just the height of $\sigma_1$ above $\sigma_2$, which in turn is just $L(\sigma_3)/L(\sigma_2)$. This leads to

$$\frac{L^{\mu\nu}(\sigma_2)}{L(\sigma_2)} = n_{\rho}(\sigma_2) \frac{L^{\rho\mu\nu}(\sigma_3)}{L(\sigma_3)}. \tag{3.14}$$

This expression will be used to obtain a relation between a sum of a two-form over a two surface and a sum of a three-form over a three-surface. A similar relation, on a smooth manifold, will involve the exterior derivative of a two-form. The essence of our definition of exterior differentiation is that it is chosen so as to mimic the usual form of Stokes' theorem.

The expression (3.14) is easily generalized to complexes of dimension greater than 3. Suppose that $\sigma_{m-1}$ is one face of $\sigma_m$ and that $\sigma_m$ is one $m$-simplex of an $m$-complex. If the unit inward normal to $\sigma_{m-1}$ is $n_{\rho}(\sigma_{m-1})$ then

$$\frac{L^{\mu_1\mu_2\cdots\mu_{m-1}}(\sigma_{m-1})}{L(\sigma_{m-1})} = n_{\mu_1}(\sigma_{m-1}) \frac{L^{\mu_1\mu_2\cdots\mu_m}(\sigma_m)}{L(\sigma_m)}. \tag{3.15}$$

This expression can be proved with techniques similar to those that led to (3.14).



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## IV. EXTERIOR DIFFERENTIATION

Consider a complex $\rho_3$ that has been subdivided into a set of three-simplexes. The integral of any two-form $A^*$ over a two-dimensional subcomplex $\rho_2$ of $\rho_3$ is defined, in a coordinate frame, as

$$I(\rho_2,\rho_3)=\sum_{\sigma_2 \text{ in } \rho_2}\int_{\sigma_2} A^*_{\mu\nu}(x^\alpha)dx^\mu\wedge dx^\nu. \tag{4.1}$$

Similarly, the integral of a three-form $B^*$ over the complex is defined as

$$I(\rho_3)=\sum_{\sigma_3 \text{ in } \rho_3}\int_{\sigma_3} B^*_{\mu\nu\tau}(x^\alpha)dx^\mu\wedge dx^\nu\wedge dx^\tau. \tag{4.2}$$

Define the quantities $A_{\mu\nu}(\sigma_2)$ and $B_{\mu\nu\tau}(\sigma_3)$ via the equations

$$A_{\mu\nu}(\sigma_2)M(\sigma_2)=\int_{\sigma_2} A^*_{\mu\nu}(x^\alpha)d^2S, \tag{4.3a}$$

$$B_{\mu\nu\tau}(\sigma_3)M(\sigma_3)=\int_{\sigma_3} B^*_{\mu\nu\tau}(x^\alpha)d^3S, \tag{4.3b}$$

with $d^iS$ and $M(\sigma_i)$ being the differential and total measures of the $\sigma_i$, respectively. The $A_{\mu\nu}(\sigma_2)$ and $B_{\mu\nu\tau}(\sigma_3)$ are the averages of their associated forms over the simplexes $\sigma_2$ and $\sigma_3$.

The relations (4.1) and (4.2) may now be rewritten as

$$I(\rho_2,\rho_3)=\sum_{\sigma_2 \text{ in } \rho_2} A_{\mu\nu}(\sigma_2)L^{\mu\nu}(\sigma_2) \tag{4.4a}$$

and

$$I(\rho_3)=\sum_{\sigma_3 \text{ in } \rho_3} B_{\mu\nu\tau}(\sigma_3)L^{\mu\nu\tau}(\sigma_3). \tag{4.4b}$$

Suppose now that the subcomplex $\rho_2$ is the boundary of $\rho_3$. Our aim is to show that $I(\partial\rho_3,\rho_3)$ may be evaluated either directly from (4.4a) or via an expression similar to (4.4b). The expression (4.4a) may be rewritten as a sum over all $\sigma_3$'s of the complex by introducing

$$J(\sigma_3)=\sum_{\sigma_2 \text{ in } \partial\sigma_3} A_{\mu\nu}(\sigma_2)L^{\mu\nu}(\sigma_2). \tag{4.5}$$

Then

$$I(\partial\rho_3,\rho_3)=\sum_{\sigma_3 \text{ in } \rho_3} J(\sigma_3), \tag{4.6}$$

since all $\sigma_2$'s on the interior of $\rho_3$ will be counted twice, each with opposite orientations, and will therefore cancel each other. Substitution of (3.14) in (4.15) and the resultant expression in (4.6) leads to

$$\begin{aligned}I(\partial\rho_3,\rho_3)&=\sum_{\sigma_2 \text{ in } \partial\rho_3} A_{\mu\nu}(\sigma_2)L^{\mu\nu}(\sigma_2)\\&=\sum_{\sigma_3 \text{ in } \rho_3}\sum_{\sigma_2 \text{ in } \partial\sigma_3} n_\rho(\sigma_2)A_{\mu\nu}(\sigma_2)\frac{L(\sigma_2)}{L(\sigma_3)}L^{\rho\mu\nu}(\sigma_3).\end{aligned} \tag{4.7}$$

This expression is greatly simplified by writing

$$A(\sigma_2)=A_{\mu\nu}(\sigma_2)L^{\mu\nu}(\sigma_2), \tag{4.8a}$$

and

$$dA(\sigma_3)=\sum_{\sigma_2 \text{ in } \partial\sigma_3} n_\rho(\sigma_2)A_{\mu\nu}(\sigma_2)\frac{L(\sigma_2)}{L(\sigma_3)}L^{\rho\mu\nu}(\sigma_3), \tag{4.8b}$$

for then

$$I(\partial\rho_3,\rho_3)=\sum_{\sigma_2 \text{ in } \partial\rho_3} A(\sigma_2)=\sum_{\sigma_3 \text{ in } \rho_3} dA(\sigma_3). \tag{4.9}$$

In this form the similarity of this expression with the usual continuum form of Stokes' theorem is quite apparent. The relation (4.8a) defines the value of the two-form $A$ on $\sigma_2$ and (4.8b) defines its exterior derivative evaluated on $\sigma_3$.

An analysis similar to that which lead to (4.8a), (4.8b), and (4.9) may be applied to complexes of dimension other than 3. Consider a complex $\rho_n$ of dimension $n$. Suppose there is defined an $m$-form $A(\sigma_m)$ on each of the $\sigma_m$'s of $\rho_n$. Thus put

$$A(\sigma_m)=A_{\mu_1\mu_2\cdots\mu_m}(\sigma_m)L^{\mu_1\mu_2\cdots\mu_m}(\sigma_m). \tag{4.10}$$

Then the exterior derivative of $A$ evaluated on $\sigma_{m+1}$ is defined as

$$dA(\sigma_{m+1})=\sum_{\sigma_m \text{ in } \partial\sigma_{m+1}} n_{\mu_0}(\sigma_m)A_{\mu_1\mu_2\cdots\mu_m}(\sigma_m)\times\frac{L(\sigma_m)}{L(\sigma_{m+1})}L^{\mu_0\mu_1\cdots\mu_m}(\sigma_{m+1}), \tag{4.11}$$

and Stokes' theorem takes the form

$$\begin{aligned}I(\partial\rho_{m+1},\rho_{m+1})&=\sum_{\sigma_m \text{ in } \partial\rho_{m+1}} A(\sigma_m)\\&=\sum_{\sigma_{m+1} \text{ in } \rho_{m+1}} dA(\sigma_{m+1}).\end{aligned} \tag{4.12}$$

If the complex consists of only one $\sigma_{m+1}$ then this expression reduces to

$$dA(\sigma_{m+1})=\sum_{\sigma_m \text{ in } \partial\sigma_{m+1}} A(\sigma_m). \tag{4.13}$$

This provides an alternative yet equivalent method for computing the exterior derivative. In some situations this expression may be more useful than (4.11). As an example it will now be shown that the value of a form, twice exterior differentiated, is zero. Consider a set of numbers $B(\sigma_{m-1})$ on the $\sigma_{m-1}$ of $\rho_{m-1}$. Suppose that each number arose as the value of an $(m-1)$-form $B$ on each of the $\sigma_{m-1}$ of $\rho_{m+1}$. The exterior derivative of $B$, evaluated on each $\sigma_m$, gives rise to another set of numbers $A(\sigma_m)$ distributed on the $\sigma_m$ of $\rho_{m+1}$. Thus

$$A(\sigma_m)=dB(\sigma_m)=\sum_{\sigma_{m-1} \text{ in } \partial\sigma_m} B(\sigma_{m-1}). \tag{4.14}$$

Now the exterior derivative of $A(\sigma_m)$ is

$$\begin{aligned}dA(\sigma_{m+1})&=\sum_{\sigma_m \text{ in } \partial\sigma_{m+1}} A(\sigma_m)\\&=\sum_{\sigma_m \text{ in } \partial\sigma_{m+1}}\sum_{\sigma_{m-1} \text{ in } \partial\sigma_m} B(\sigma_{m-1}).\end{aligned}$$

However each $\sigma_{m-1}$ is counted twice, each time with opposite orientations, thus

$$dA(\sigma_{m+1})=d\,dB(\sigma_m)=0.$$

Exactly the same result occurs in the continuum theory of differential forms.

As another example consider the flux of a constant vector $A$, with components $A^\mu$, over the surface of one $m$-sim-



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plex in an $m$-complex. Clearly this quantity vanishes and is expressed as

$$0=\sum_{\sigma_{m-1}\text{in }\partial\sigma_m} A^{\mu}n_{\mu}(\sigma_{m-1})L(\sigma_{m-1}).$$

As this expression is true for any constant field $A$ it follows that

$$0=\sum_{\sigma_{m-1}\text{in }\partial\sigma_m} n_{\mu}(\sigma_{m-1})L(\sigma_{m-1}).$$

Unfortunately, since $n_\mu$ need not be continuous across each $\sigma_{m-1}$, this expression cannot be applied directly to complexes of more than one $m$-simplex. However after a contraction with $L^{\mu\mu_1\mu_2\cdots\mu_{m-1}}(\sigma_m)$ and using (3.15) this expression reduces to

$$0=\sum_{\sigma_{m-1}\text{in }\partial\sigma_m} L^{\mu_1\mu_2\cdots\mu_{m-1}}(\sigma_{m-1}),$$

which is easily extended to complexes, thus

$$0=\sum_{\sigma_{m-1}\text{in }\partial\rho_m} L^{\mu_1\mu_2\cdots\mu_{m-1}}(\sigma_{m-1}).$$

This expression can also be proved directly from the definition (3.9).

## V. THE CONTINUUM LIMIT

The definitions (4.10) and (4.11) may be extended to complexes built from blocks other than simplexes. For example, an initial manifold could be constructed by piecing together a sequence of three-dimensional cubes. Each such cube could be subdivided, by the addition of extra vertices, legs, and faces, into a set of three-simplexes thus producing a three-complex. To this complex the identity (4.12) would apply. However the terms of this expression may be regrouped so that those terms involving the faces of the three-simplexes are combined into terms involving the faces of the cubes. Similarly the terms involving the three-simplexes would be grouped into terms involving the cubes. In effect the expression (4.12) is unaltered except that the objects in the summation are now parallelograms and parallel cubes instead of triangles and tetrahedrons.

To show that the similarity of (4.9) and (4.12) with Stokes' theorem is not just a consequence of formal algebraic manipulations, the nature of (4.9), over a sequence of complexes, will now be investigated. The following assumptions are necessary.

(i) The dimension of the complex is 3. A similar analysis may be used for higher dimensions.

(ii) The sequence of complexes converges, as the number of $\sigma_3$'s is increased without limit while keeping the total measure, fixed, to a smooth differentiable manifold.

(iii) The $\sigma_3$'s of each complex are sufficiently small that the values of $A^{*}_{\mu\nu}$ on the faces $\sigma_2$ of $\sigma_3$ may be derived from a Taylor series based at some point within $\sigma_3$. Thus

$$A^{*}_{\mu\nu}(x^\alpha)=A^{*}_{\mu\nu}+A^{*}_{\mu\nu,\rho}\delta x^\rho+O(\delta x)^2, \qquad (5.1)$$

with $\delta x^\alpha=x^\alpha-x^\alpha_0$, and $x^\alpha_0$ is the point, within $\sigma_3$, from which the Taylor series is developed.

(iv) All $\sigma_3$'s are three-cubes (parallelepipeds).

For a cube the inward pointing normals for two opposite faces are equal apart from their directions. Thus the terms in (4.8b) may be regrouped as

$$dA(\sigma_3)=\sum_{\substack{\text{three adjacent}\\ \text{faces}}} n_\rho(\sigma_2)\Delta A_{\mu\nu}(\sigma_2)\frac{L(\sigma_2)}{L(\sigma_3)}L^{\rho\mu\nu}(\sigma_3). \qquad (5.2)$$

However, from (4.3a)

$$\Delta A_{\mu\nu}(\sigma_2)L(\sigma_2)=\int_{\sigma_2-\sigma_{2*}} A^{*}_{\mu\nu}(x^\alpha)d^2S, \qquad (5.3)$$

with $\sigma_{2*}$ being the face opposite $\sigma_2$. Substitution of (5.1) into (5.3) and noting that $A^{*}_{\mu\nu}$ and $A^{*}_{\mu\nu,\rho}$ are constant throughout $\sigma_3$ results in

$$\Delta A_{\mu\nu}(\sigma_2)L(\sigma_2)=A^{*}_{\mu\nu,\rho}\int_{\sigma_2-\sigma_{2*}}\delta x^\rho\, d^2S.$$

By projecting $\delta x^\rho$ onto the normal and tangential vectors of $\sigma_2$ it is not hard to show that this last integral equals $n^\rho L(\sigma_3)$. Thus

$$\Delta A_{\mu\nu}(\sigma_2)L(\sigma_2)=A^{*}_{\mu\nu,\rho}n^\rho(\sigma_2)L(\sigma_3)$$

and consequently (4.8b) becomes

$$dA(\sigma_3)=A^{*}_{\mu\nu,\rho}\sum_{\substack{\text{three adjacent}\\ \text{faces}}} n_\alpha(\sigma_2)n^\rho(\sigma_2)L^{\alpha\mu\nu}(\sigma_3)\,.$$

But from (3.10a) and (3.10b) the summation reduces to $L^{\rho\mu\nu}(\sigma_3)$. Thus

$$dA(\sigma_3)=A^{*}_{\mu\nu,\rho}(\sigma_3)L^{\rho\mu\nu}(\sigma_3)$$

and (4.7) becomes

$$I(\partial\rho_3,\rho_3)=\sum_{\sigma_2\text{in }\partial\rho_3} A_{\mu\nu}(\sigma_2)L^{\mu\nu}(\sigma_2)$$

$$=\sum_{\sigma_3\text{in }\rho_3} A_{\mu\nu,\rho}(\sigma_3)L^{\rho\mu\nu}(\sigma_3), \qquad (5.4)$$

with $A_{\mu\nu,\rho}(\sigma_3)=A^{*}_{\mu\nu,\rho}$.

This last result shows clearly that this definition of exterior differentiation does reduce to the usual form when the Regge manifold and the forms built on it are made smooth and differentiable.

## VI. CONCLUSION

It has been shown that the concept of exterior differentiation has a natural extension to the Regge calculus. Results similar to (4.9) may be found in references 8–10. The motivation for the development of a Regge version of exterior differentiation arises in the attempt to show that certain Regge expressions "converge" to their usual classical counterparts under certain conditions. An example of this process, that our version of Stokes' theorem reduces to its usual form when applied to differtiable forms, has been presented in Sec. V. A more ambitious project would be to prove (or disprove) that the Regge field equations reduce to the Einstein field equations when an appropriate limiting process is applied. This may form the basis of a future investigation.

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