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THE RIEMANN AND EXTRINSIC CURVATURE TENSORS IN THE REGGE CALCULUS

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A detailed analysis of the Riemann tensor in the neighbourhood of one bone and of the extrinsic curvature in the neighbourhood of one triangular face in a simplicial geometry is presented. Unlike most previous analyses this analysis makes no reference to any particular choice of smoothing scheme. Explicit formulae will be presented for both the Riemann and extrinsic curvature tensors. These results are applied, using the formalism developed in an earlier paper [1], in deriving an exact formula for the integral extrinsic curvature. It is argued that for integrals of $R^2, R^3 \cdots$ contributions must be expected from the legs, vertices,... rather than just from the bones. This work provides the background material for the following paper in which the Gauss-Codacci equation is applied to a Regge spacetime.

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1. Introduction.

This is the first of two papers dealing with certain aspects of the Riemann and extrinsic curvature tensors on a Regge spacetime. The basic aim is to produce a "3+1" formulation of the Regge calculus.

In an earlier paper [1] a first order continuous time formulation of the Regge calculus was developed. Unfortunately the evolution equations presented in that paper made reference to the defects in the full 4-dimensional spacetime. If a "3+1" formulation is to be achieved then it will be necessary to develop an expression for those defects in terms of the defects on each leaf and of the extrinsic curvature of the foliation. It is our aim to present such a relationship in this pair of papers.

In this paper the basic expression for the Riemann and extrinsic curvature tensors will be presented. In the following paper these results will be employed in the development of the Gauss-Codacci equation. It is from this expression that the relation linking the 4-defects with the 3-defects will be obtained.

Many authors [2,3,4,5,6,7,8,9] have previously provided derivations of the Riemann tensor for simplicial geometries. In each of these papers the authors made one or more of the following assumptions,

i) that the simplicial geometry may be explicitly approximated by a parameterized family of smooth geometries,

ii) that the final expression for the Riemann tensor does not depend upon the choice of that family,

iii) that the defect for a finite loop equals the sum of the defects for each of the infinitesimal loops that make up that loop and

iv) that there is no interaction amongst neighbouring bones.

The last three assumptions are, arguably, the most important. The second assumption is necessary when an explicit choice has been made for the smooth metric. It is quite possible that the global properties of a singular metric may well depend upon the internal structure of the singularity (see [10]). The third assumption arises when formulae familiar from continuum differential geometry are employed. For example, the change in a vector when parallel transported around an infinitesimal loop is just a small rotation of that vector in some plane. From this it is argued that the Riemann tensor is proportional to the defect for this infinitesimal loop. However in the discrete geometry the defect must be calculated for a finite loop drawn in the far flat regions. It is at this point that the third assumption is made. The fourth assumption, that there is no interaction between neighbouring bones, allows one to write, for example, the Hilbert action as a sum over each of the bones. It will be shown in the last section § 6 that interactions between neighbouring bones must be expected.

The fundamental result, that the Hilbert and Regge actions are equal, can be no more valid than each of the above assumptions. It is therefore important to try to remove some of these assumptions. In this paper the last three assumptions will not be required.

There are only a handful of published papers [11,12,13,14,15,16] in which the extrinsic curvature tensor for a simpilcial spacetime is discussed. The works of Porter [11,12] and that of Williams [13] contain some approximate expressions for particular components of $K_{\mu\nu}$. They obtain their expressions by identifying certain terms in the Regge field equations. They do not, however, provide any general expression for all of the components of $K_{\mu\nu}$. Such an expression is provided by Piran and Williams [14] and also by Friedmann and Jack [15]. The method employed in both of these papers is based upon the construction of analogies between the structure of certain tensor fields on a smooth spacetime and of those same fields on a Regge spacetime. They first develop an analogy for the metric tensor. They then develop analogies for the first time derivative of the metric, the lapse function and the shift vectors. This leads directly to an expression for the $K_{\mu\nu}$.

The main difficulty with this approach is that there is no unique construction for the shift vectors. Its not surprising then that the results of the two papers are not identical. In this paper a well defined expression for the $K_{\mu\nu}$ will be obtained by applying the usual formulae to a family of smooth spacetimes.

2. Preliminaries.

Consider a general discrete time "3+1" Regge spacetime [1,17] and focus attention on the 4-dimensional region surrounding one timelike bone. This will consist of the set of worldtubes adjacent to the bone and the (portions of the) pair of leaves that these worldtubes join. Each worldtube has a tetrahedral cross section and their intersections with the two leaves are just the various tetrahedra of those leaves. In the following analysis this sub-region will be treated as if it where the whole spacetime.

Our aim is to compute the Riemann and extrinsic curvature tensors for this simple spacetime. However as the metric and the embedding of the leaves in the spacetime need not be smooth one can not apply the usual formulae. The approach to be adopted here is the usual technique of applying the formulae to a continuous family of smooth spacetimes and then developing a suitable limiting procedure. It will therefore be necessary to view the final results as distributions rather than as ordinary functions.

Let M represent the spacetime, $T_i, i = 1, 2, 3...$ the various worldtubes and $S^\uparrow, S^\downarrow$ the upper and lower leaves. The various tetrahedra of the leaves will be denoted by ${s^\uparrow}_i, {s^\downarrow}_i, i = 1, 2, 3....$ Suppose that ${s^\downarrow}_i$ and ${s^\downarrow}_j$ are a pair of adjacent tetrahedra in $S^\downarrow$. Then their triangular interface will be denoted by ${s^\downarrow}_{ij}$. The timelike bone will be represented by σ. Its image in $S^\downarrow$ will be denoted by σ' and will represent the spacelike bone of $S^\downarrow$ (this is also the spacelike leg in $S^\downarrow$ upon which σ stands). (Notice that these definitions differ from those used in Brewin [1,17] .)

3. The metric.

Let the smooth family of metrics be denoted by $g_{\mu\nu}$ and let the associated smoothing parameter be $\#_g$. The limiting form of $g_{\mu\nu}$ as $\#_g \to 0$ will be the original discrete metric of M. Now in any Regge spacetime any vector that is parallel to the bone will suffer no change when parallel transported around any loop. The $g_{\mu\nu}$ will be chosen so as to preserve this symmetry. Thus it is always possible to choose two vectors, p^μ and q^μ, such that

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + p_\mu p_\nu - q_\mu q_\nu \tag{3.1a}$$

with

$$p^\mu{}_{;\nu} = q^\mu{}_{;\nu} = 0\ . \tag{3.1b}$$

The two dimensional metric $\overline{g}_{\mu\nu}$ is the metric of the 2-dimensional sheet that is perpendicular to the bone σ. This sheet will be denoted by C. The metric on C is of Euclidian signature and will, as $\#_g \to 0$, be flat everywhere except at the point where C intersects the bone. The vectors p^μ and q^μ will be chosen so that p^μ is parallel to σ'.

The "3+1" decomposition of the metric is normally written as

$$g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu \tag{3.2a}$$

where n^μ is the unit timelike normal to $S^\downarrow$ and $h_{\mu\nu}$ is the induced (smooth) 3-metric on $S^\downarrow$. Notice that n^μ may be smoothed independently of $g_{\mu\nu}$. The associated smoothing parameter will be represented by $\#_n$. Notice also that since $n_\mu = g_{\mu\nu} n^\nu$ it will be necessary to view the n_μ as being dependent on both $\#_g$ and $\#_n$. Now since p^μ is parallel to σ' it is possible to write

$$h_{\mu\nu} = \overline{h}_{\mu\nu} + p_\mu p_\nu \tag{3.2b}$$

for some 2-metric $\overline{h}_{\mu\nu}$. This 2-metric represents the metric of some 2-dimensional sheet which, in general, will not coincide with C. This sheet will be represented by C'.

4. The Riemann tensor.

The form of the metric in (3.1a,b) leads to an immediate simplification in the computation of the Riemann tensor, namely

$$R^{\mu}{}_{\nu\alpha\beta}(g) = R^{\mu}{}_{\nu\alpha\beta}(\overline{g}) \, .$$

Now consider a small rectangle, drawn in C, and generated by the two vectors δx_1^μ and δx_2^μ. Suppose v^μ is any vector lying in the tangent space to C. When v^μ is parallel transported around this loop the net change in v^μ will be equivalent to a rotation of v^μ through some small angle – the defect angle. Denote this angle by $\delta\alpha$. It follows that

$$\delta v^\mu = \left(\frac{\delta\alpha}{\delta A}\right) U^\mu{}_\nu \, v^\nu \, \delta A$$

where $U^{\mu\nu}$ is the bivector of C and δA is the area of the loop. Now by writing

$$\delta A = U_{\alpha\beta} \, \delta x_1^\alpha \, \delta x_2^\beta$$

it is easy to see that the components of the Riemann tensor are given by

$$R^{\mu}{}_{\nu\alpha\beta}(g) = d(g) U^\mu{}_\nu U_{\alpha\beta} \tag{4.1}$$

where $d(g) = \delta\alpha/\delta A$ is the defect per unit area on C. This is the fundamental form of the Riemann tensor for one timelike bone. Quite clearly the limit $\lim_{\#_g \to 0} d(g)$ does not exist as a well defined ordinary function (the limit diverges for points on σ yet it vanishes everywhere else). The correct approach would be to view the limit as a generalized function. Thus $\lim_{\#_g \to 0} d(g)$ will be viewed as a Dirac distribution, with respect to the measure $\sqrt{-g}$, with compact support on σ.

An important property of the defects for various small loops on C is that they are additive. That is, the net defect for a loop composed of a number of smaller loops is just the sum of the defects for the individual loops. This may be proved as follows. Denote the loop by $\mathcal{L}$ and let this loop be composed of the smaller loops $\mathcal{L}_i, i = 1, 2, 3....$ Suppose the defects associated with these loops are θ and θ_i respectively. Similarily let $U^{\mu\nu}$ and $U^{\mu\nu}{}_i$ be the associated bivectors. The Bianchi identities for this set of loops may be written as

$$\exp(\theta U^\mu{}_\nu) = \prod_i \exp(\theta U^\mu{}_\nu)_i \,.$$

But since each of the components of $U^{\mu\nu}$ and $U^{\mu\nu}{}_i$ are equal (which may be proved by noting that $U^{\mu\nu}$ is tangent to C and $p^\mu{}_{;\nu} = q^\mu{}_{;\nu} = 0$ thus $U^{\mu\nu}{}_{;\alpha} = 0$) it is clear that the Bianchi identities may be reduced to

$$\theta = \sum_i \theta_i$$

which proves the assertion.

Now imagine that the loop $\mathcal{L}$ is chosen as some large loop in the almost flat far regions of C and that each of the $\mathcal{L}_i$ is an infinitesimal loop. Then the θ of $\mathcal{L}$ will, as $\#_g \to 0$, be just the defect α of the original discrete spacetime M. Each of the θ_i may be accurately approximated by $d(g)\delta^2 A_i$ where $\delta^2 A_i$ is the area of the loop $\mathcal{L}_i$. Thus it follows that the above relation may be reduced, as $\#_g \to 0$, to

$$\alpha = \lim_{\#_g \to 0} \int_C d(g)\sqrt{g}\, d^2x \,. \qquad (4.2)$$

This result establishes the link between the global defect of the discrete geometry and the local defect of the smooth geometry. Combining (4.1) and (4.2) leads directly to

$$\lim_{\#_g \to 0} \int_M R\sqrt{-g}\, d^4x = 2\alpha A \qquad (4.3)$$

where A is the area of σ. This particular result is not new but the method employed, which avoids the use of any specific smoothing schemes, is new (cf. [5]).

It is rather easy to see that this same method may also be applied to any spacelike bone. However in this instance the geometry of C is hyperbolic rather than Euclidian.

5. The Extrinsic curvature tensor.

The extrinsic curvature tensor is normally defined by

$$K^{\mu}{}_{\nu} = n^{\mu}{}_{;\nu} \tag{5.1}$$

where n^{μ} is the unit tangent vector to the geodesics normal to $S^{\downarrow}$. On the interior of any one tetrahedron the $K^{\mu}{}_{\nu}$ must vanish. However at the various interfaces between neighbouring tetrahedra the $K^{\mu}{}_{\nu}$ when viewed as ordinary functions must be singular. By introducing a smoothing scheme for the embedding of $S^{\downarrow}$ in M it is possible to re-interpret $K^{\mu}{}_{\nu}$ as a generalized function in the neighbourhoods of the triangular interfaces. However the behaviour of the $K^{\mu}{}_{\nu}$ in the neighbourhood of the bone σ' of $S^{\downarrow}$ depends crucially upon the smoothing scheme. Consequently the following analysis will apply only to the neighbourhood of a typical triangular interface in $S^{\downarrow}$.

Consider a typical pair of adjacent tetrahedra $s^{\downarrow}{}_1$ and $s^{\downarrow}{}_2$ of $S^{\downarrow}$. Denote the triangular interface between this pair by $s^{\downarrow}{}_{12}$. Now consider some arbitrary path $\Gamma(w)$, parameterized by the proper distance w, joining some point P in $s^{\downarrow}{}_1$ to some point Q in $s^{\downarrow}{}_2$. Along Γ the normal n^{μ} is a well defined vector field varying smoothly with w. If n^{μ} at P is parallel transported to some intermediate point on Γ then it will be related to the actual n^{μ} at that point by a rotation, through some angle β, in a plane perpendicular to $s^{\downarrow}{}_{12}$. The angle β will, when both $\#_g$ and $\#_n$ are sufficiently small, depend only upon the distance measured away from the interface $s^{\downarrow}{}_{12}$ (ie. in the discrete space n^{μ} changes only upon crossing the interface).

The change in n^{μ} arising from the parallel transportation of n^{μ} along Γ will, therefore, be given by

$$\delta n^{\mu} = (\delta\beta) W^{\mu}{}_{\nu} n^{\nu} = n^{\mu}{}_{;\nu}\, \delta x^{\nu}$$

where $W^{\mu\nu}$ is the bivector of the 2-dimensional plane in $S^{\downarrow}$ and perpendicular to $s^{\downarrow}{}_{12}$. However

$$\delta\beta = \frac{\delta\beta}{\delta s}\, t_{\alpha}\, \delta x^{\alpha}$$

where s is the proper distance measured along the geodesic normal to ${s^\downarrow}_{12}$ in $S^\downarrow$ and t^α is the unit tangent to that geodesic. Combining this expression, the previous expression and (5.1) leads to

$$K^\mu{}_\nu = \left(\frac{\delta\beta}{\delta s}\right) W^\mu{}_\alpha n^\alpha t_\nu \ .$$

By choosing

$$W^\mu{}_\nu = n^\mu t_\nu - t^\mu n_\nu$$

as a representation for $W^\mu{}_\nu$ the previous expression may be reduced to

$$K^\mu{}_\nu = \left(\frac{\delta\beta}{\delta s}\right) t^\mu t_\nu \ . \tag{5.2}$$

This is the fundamental form of the extrinsic curvature tensor for a triangle ${s^\downarrow}_{jk}$ in $S^\downarrow$. As with $d(g)$, the quantity $\delta\beta/\delta s$ is best viewed as a Dirac distribution with support on ${s^\downarrow}_{jk}$ as $\#_g, \#_n \to 0$.

Two important steps remain, firstly to identify the strength of this Dirac distribution and, secondly to then express that strength in terms of the Cauchy data of a "3+1" formulation. The first step will be achieved by evaluating a volume integral of $K = K^\mu{}_\mu$. For the second step the formalism developed in a recent paper [17] will be used.

The trace of the extrinsic curvature is

$$K = \frac{\delta\beta}{\delta s} \ . \tag{5.3}$$

Now choose any 3-dimensional region Ω_{jk} in $S^\downarrow$ that contains ${s^\downarrow}_{jk}$ but does not contain any points of σ'. The integral of K throughout this region is

$$\int_{\Omega_{jk}} K\sqrt{h}\, d^3x = \int_{\Omega_{jk}} \frac{\delta\beta}{\delta s}\sqrt{h}\, d^3x \ .$$

The volume element in the right hand integral may be written as dsd^2B where d^2B is the element of area of ${s^\downarrow}_{jk}$ and s is the distance measured normal to ${s^\downarrow}_{jk}$. As $\#_g, \#_n$ are reduced to zero the integral in ds is constant over ${s^\downarrow}_{jk}$.

Thus one obtains

$$\begin{aligned}\lim_{\#_g,\#_n\to 0}\int_{\Omega_{jk}} K\sqrt{h}\,d^3x &= \lim_{\#_g,\#_n\to 0}\int_{\Omega_{jk}}\left(\frac{\delta\beta}{\delta s}ds\right)d^2B\\ &= \lim_{\#_g,\#_n\to 0}\int_{s^{\downarrow}{}_{jk}}\Delta\beta\, d^2B\\ &= \Delta\beta\, B \qquad (5.4)\end{aligned}$$

where B is the area of $s^{\downarrow}{}_{jk}$ and $\Delta\beta$ is the angle between the two normals on either side of $s^{\downarrow}{}_{jk}$. It is not possible to extend this result to the whole spacetime without first investigating the behaviour of K on σ'. This will be deferred until section § 6. The result will be

$$\lim_{\#_g,\#_n\to 0}\int_{S^{\downarrow}} K\sqrt{h}\,d^3x = \sum_{\sigma_2(S^{\downarrow})}\Delta\beta\, B \qquad (5.5)$$

in which the summation extends over all of the triangles in $S^{\downarrow}$.

This result has also been obtained by Hartle and Sorkin [16] . In their approach they choose to construct a Regge action which would be additive (ie. $I[g_1+g_2]=I[g_1]+I[g_2]$ for two spacetimes g_1 and g_2 that are identified on one leaf). By comparing their action with the usual (additive) Hilbert action they where led to (5.5).

In any discrete space the value of $\Delta\beta$ would be computed by way of some suitable trigonometric formulae. However in the case where time is treated as a continuous variable it is possible to develop an explicit expression for the $\Delta\beta$'s. The approach to be developed here is a direct generalization of a method used by Brewin [1] . The notation in the following analysis draws heavily upon that paper, thus for simplicity some of the basic notation and results of that paper are summarized here.

The lapse functions are defined on each vertex and happen to appear in the theory only as their averages on each leg. Consider one leg L_i and suppose that its vertices have been labelled (1) and (2). Let N_{1i} and N_{2i} be the lapse functions defined on the two vertices. Then the average lapse on L_i is just $N_i=(N_{1i}+N_{2i})/2$. Suppose that the angles between the worldlines of these vertices and the leg are denoted by γ_{1i} and γ_{2i}. Then a consequence of one of the field equations is that $\gamma_{1i} = \gamma_{2i}$. This common value will be denoted by γ_i. The only field equation that will be needed here is just

$$\frac{dL_j}{d\lambda} = 2N_j \sinh \gamma_j \tag{5.6}$$

where λ is the time parameter.

Consider the pair of adjacent tetrahedra ${s^\downarrow}_1$ and ${s^\downarrow}_2$. The associated pair of worldtubes are T_1 and T_2. The 3-dimensional region common to T_1 and T_2 is T_{12} and is the worldtube generated by s_{12}. Let β_1 be the angle from s_1 to T_{12} and let β_2 be the angle from s_2 to T_{12}. Then the value of $\Delta\beta$ for s_{12} is just $\beta_1 + \beta_2$. Therefore it is only necessary to consider the detailed computation of the various β_i defined within one T_i.

Suppose that the leaf $S^\uparrow$ is located only a short distance away from $S^\downarrow$. The worldtube T_1 is therefore short, its height being of order $\Delta\lambda$. Consider now the application of Stoke's theorem to T_1. Let m_i^μ be the various (outward pointing) spacelike normals to the T_{1i} of T_1 and let V_i be the 3-volumes of these faces. Furthermore, let $V^\uparrow$ and $n^{\uparrow\mu}$ be the volume and (outward pointing) normal to ${s^\uparrow}_1$. The $V^\downarrow$ and $n^{\downarrow\mu}$ are defined similarily. Then Stoke's theorem leads to

$$0 = V^\uparrow n^{\uparrow\mu} + V^\downarrow n^{\downarrow\mu} + \sum_i V_i m_i^\mu \,.$$

This expression will, in the next few lines, be contracted with $n^{\uparrow\mu}$ but first observe that

$$n^{\uparrow\mu} {n^\downarrow}_\mu = 1 - n^{\downarrow\mu} \frac{d{n^\downarrow}_\mu}{d\lambda} \Delta\lambda + O(\Delta\lambda)^2$$

and

$$\sinh \beta_i = {n^\uparrow}_\mu m_i^\mu \,.$$

(A definition of the hyperbolic angles may be found in [17].) Now since $n^{\downarrow\mu} {n^\downarrow}_\mu = -1$ it follows that

$$0 = n^{\downarrow\mu} \frac{d{n^\downarrow}_\mu}{d\lambda}$$

and consequently the contracted form of the above equation may be reduced to

$$0 = -\Delta V + \sum_i \Delta V_i \sinh \beta_i + O(\Delta\lambda)^2$$

where $\Delta V = V^{\uparrow} - V^{\downarrow}$ and ΔV_i is the small value of V_i. The ΔV_i will be computed, accurate to $O(\Delta\lambda)^2$, as the product of the area of the base ${s^{\downarrow}}_{1i}$ with the average height of T_{1i}. Denote by $\Delta h(x)_i$ the distance measured from the point x in s_{1i} along the geodesic normal to s_{1i} in T_{1i} (see Fig(1)). Let $\overline{\Delta h}_i$ be the average of $\Delta h(x)_i$ over s_{1i} and let B_i be the area of s_{1i}, then

$$\Delta V_i = B_i \overline{\Delta h}_i + O(\Delta\lambda)^2$$

Now since $\Delta h(x)_i$ is a linear function over s_{1i} its average may be expressed as

$$\overline{\Delta h}_i = \frac{1}{3} \sum_j \overline{\Delta h}_{ij}$$

where $\overline{\Delta h}_{ij}$ is the average of $\Delta h(x)_i$ over each of the legs of s_{1i} and the sum includes each leg of s_{1i}. The $\overline{\Delta h}_{ij}$ associated with the leg L_j of T_{1i} can be computed by projecting $\overline{\Delta h}_{ij}$ onto the timelike face generated by L_j. Thus if ϕ_{ij} is the angle from this face to the base s_{1i} then

$$\overline{\Delta h}_{ij} = N_j \cosh \gamma_j \cosh \phi_{ij} \Delta\lambda\,.$$

In reference [1] it was shown that

$$\sinh \phi_{ij} = \frac{2}{L_j} \frac{\partial B_i}{\partial L_j} \tanh \gamma_j$$

from which the $\cosh \phi_{ij}$ may be easily obtained. Upon combining the above expressions one finds that

$$\Delta V = \sum_i \sum_{j(i)} \frac{B_i}{3} N_j \cosh \gamma_j \cosh \phi_{ij} \sinh \beta_i \Delta\lambda$$

where the first sum includes each face of $s^{\downarrow}{}_1$ and the second sum includes only those legs of that face. However it is also true that

$$\begin{aligned}\Delta V &= \sum_i \frac{dL_i}{d\lambda}\frac{\partial V}{\partial L_i}\Delta\lambda \\ &= \sum_i 2N_i \sinh\gamma_i \frac{\partial V}{\partial L_i}\Delta\lambda\,.\end{aligned}$$

It is important to remember that the N_i are the averages of the lapse functions on the pairs of vertices of each leg. Thus in equating coefficients in each of these expressions (since they are identical for any value of the lapse functions and the γ's, ϕ's and β's do not depend on the lapse functions) one must first write, for example on the leg (12), $N_i = (N_{1i} + N_{2i})/2$. After making this substitution for each of the N_i, gathering together the terms in each of the N_{ij} associated with the vertex (i) and then comparing terms one will obtain

$$\sum_{j(i)} \sinh\gamma_j \frac{\partial V}{\partial L_j} = \sum_{j(i)}\sum_{k(j)} \frac{B_k}{6}\cosh\gamma_j \cosh\phi_{kj} \sinh\beta_k\,. \tag{5.7}$$

The two summations over $j(i)$ includes only those legs attached to vertex (i) while the summation over $k(j)$ includes only the pair of triangles attached to the j^{th} leg of $s^{\downarrow}{}_1$. This represents a set of four equations (one for each vertex (i) of $s^{\downarrow}{}_1$) for the four unknown β_k. By assembling the above equations into a matrix form and noting that the entries depend only upon the (independent) γ_i and L_i it is easy to see that the above equations do possess a unique solution. Upon solving for the β_i and then computing the $\Delta\beta$ one will obtain a complete specification of K (and consequently also of $K^{\mu}{}_{\nu}$) in terms of the basic data, namely the L's, N's and γ's.

6. General spacetimes.

The typical Regge spacetime will not be as simple as that considered in the preceding analysis. Generally the Regge spacetime M will consist of a number of timelike and spacelike bones. These bones will not normally be isolated since on each leg (ie. a *knuckle*) there may well be attached more than bone. As it has already been shown that there is a contribution to the scalar curvature from the bones it is tempting to ask if there is any similar contribution from the knuckles. Such contributions might arise when calculating, for example, higher derivative actions. Although the exact behaviour of the curvature tensor in the region of any one knuckle can not be calculated by the methods of the previous sections it is possible to give an overview of the expected behaviour. In this section some brief speculative ideas will be presented.

For any spacetime the Riemann tensor may always be diagonalized in the form [18]

$$R^{\mu}{}_{\nu\alpha\beta} = \sum_{i=1}^{6} \rho_i \left(U^{\mu}{}_{\nu} U_{\alpha\beta}\right)_i \tag{6.1}$$

for some set of (complex) scalar functions ρ_i and some set of (complex) bivectors $U^{\mu\nu}{}_i$. Comparing this with the general form (4.1) suggests that in the neighbourhood of any one bone all but one of the ρ_i will be essentially zero and the associated bivector will be perpendicular to the bone. However in the vicinity of the knuckles all that can be said of the ρ_i is that they are smooth well behaved functions when $\#_g, \#_n$ are non-zero. This behaviour would reflect the interaction of the various bones attached to this knuckle.

Consider now the integral of R^2 throughout M. This may be written as

$$\int_M R^2 \sqrt{-g}\, d^4x = \int_M \sum_{i,j=1}^{6} \rho_i \rho_j \sqrt{-g}\, d^4x\, .$$

Consider, as an example, just one knuckle and the set of bones attached to it. Let m be the number of bones attached to the knuckle. Now consider the region of M in which the $\rho_i \neq 0$. This region may be partitioned into the regions M' and $M'_j, j = 1, 2, 3 \cdots m$ chosen so that M' is a thin tube enclosing the knuckle and the M'_j are thin slabs that enclose the j^{th} bone on the knuckle. The above integral may then be rewritten as

$$\int_M R^2 \sqrt{-g}\, d^4x = \int_{M'} \sum_{i,j=1}^{6} \rho_i \rho_j \sqrt{-g}\, d^4x + \sum_{j=1}^{m} \int_{M'_j} \sum_{i=1}^{6} \rho_i^2 \sqrt{-g}\, d^4x.$$

Let $\overline{\rho}$ be an estimate of the average of all of the ρ_i over all of the points in M'. Similarily, let $\overline{\rho_j}$ be an estimate of the average of all of the ρ_i throughout M'_j. Let $V(M')$ and $V(M'_j)$ be the 4-volumes of M' and M'_j respectively. Then it follows that as $\#_g \to 0$

$$V(M') \sim L\#_g^3$$
$$V(M'_j) \sim A_j\#_g^2$$

where L is the length of the knuckle and A_j is the area of the bone inside M'_j. Since the volume integral of R is known to be finite as $\#_g \to 0$ it follows that

$$\overline{\rho_j} \sim \#_g^{-2}$$

for each bone. Suppose that on the knuckle

$$\overline{\rho} \sim \#_g^{-n}$$

for some unknown integer n. Using these estimates the above relation may be estimated as

$$\int_M R^2 \sqrt{-g}\, d^4x \sim L \cdot \#^{-2n} \cdot \#^3 + \sum_{j=1}^{m} A_j \cdot \#^{-4} \cdot \#^2\, .$$

If the right hand side is to remain finite as $\#_g \to 0$ then $n = 2.5$. A similar analysis could be used in estimating the behaviour of the ρ_i for higher powers of R. For R^3 it would be necessary to expect contributions from the vertices as well as the knuckles and bones of M. Since $n = 2.5$ is necessary if the volume integral of R^2 is to be finite it follows that for the volume integral of R the contribution from the knuckle behaves like $\#^{1/2}$ and thus vanishes as $\#_g \to 0$. A direct consequence of this result is that for any well behaved function f over M

$$\lim_{\#_g \to 0} \int_M f(x) R \sqrt{-g}\, d^4x = 2 \sum_{\sigma(M)} (\overline{f} \alpha A)_\sigma$$

where $\overline{f}$ is the average of f over a bone and the summation extends over all of the bones in the spacetime. This shows that for the Hilbert action integral, in which $f = 1$ throughout M, one may proceed as if there were no interactions between neighbouring bones. This result has been used by many other authors but without consideration of the contributions from the knuckles and vertices.

A similar diagonalization may be used for the $K^\mu{}_\nu$. Thus

$$K^\mu{}_\nu = \sum_{i=1}^{3} \tau_i \left(t^\mu t_\nu\right)_i \tag{6.2}$$

where the t^μ_i are a particular set of unit orthogonal spatial vectors. In the vicinity of any triangle all but one of the τ_i will be essentially zero and the associated t^μ_i will be perpendicular to that interface. It has already been argued that the functional form of the τ_i in the vicinity of the interfaces between the triangles (ie. the knuckles) may be rather complicated. However there are some important non-linear combinations of the $K^\mu{}_\nu$ that will require a detailed knowledge of the τ_i in these regions. For example the combination

$$\left(K^\mu{}_\mu\right)^2 - K^\mu{}_\nu K^\nu{}_\mu = \sum_{i \neq j} \tau_i \tau_j$$

is essentially non-zero only on the knuckles. Presumably this expression may be interpreted as some distribution with support on the legs of the leaves. Such an interpretation may be presented in a future paper.

Using techniques similar to that just presented it is not hard to show that contributions to the integral of, for example, K^2 may arise from the knuckles. If that integral is finite then it follows that there will be no contributions from the knuckles in computing the integral of K

and that the simple result (5.4) may be immediately generalized to the result (5.5) of Hartle and Sorkin [16].

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Figures

Fig 1. The timelike face T_{1i} of the worldtube T_1. The dotted line on the interior of T_{1i} is a segment of the geodesic normal to the base (123) and passing through the mid-point of (12).