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MARGINALLY TRAPPED SURFACES IN A SIMPLICIAL SPACE.

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A simple expression for the local construction of a marginally trapped surface in a simplicial spacetime will be presented. The result will be obtained by two distinct methods; in one method a differential equation will be solved, in the other method the aberration formula from special relativity will be used.

1. Introduction.

In any spacetime the existence and location of black-holes can only be determined by a global analysis (ie. one must construct all past-directed non-spacelike curves from future null-infinity [1]). This procedure cannot be applied directly to a numerical spacetime since such spacetimes are usually not evolved to null-infinity. However, it is possible to infer the existence of a black-hole from the existence of a closed marginally trapped surface. The advantage of this approach is that the marginally trapped surface can be determined by a local analysis (ie. within one Cauchy surface).

The essential idea is to find a closed 2-dimensional surface for which the outgoing null geodesics have vanishing divergence. The procedure may be broken into two steps. First, one determines locally the form of all 2-surfaces for which the divergence vanishes. Second, one attempts to construct a closed 2-surface by piecing together the local 2-surfaces. This step is also the crucial step. In any space it is always possible to find, locally, a divergence free 2-surface (eg. any 2-plane in flat space). However, only when a black-hole exists can a globally divergence free 2-surface be found [1] .

The main result, equation (2.6), will be obtained by two distinct methods. In the first method the problem is viewed as one of finding an appropriate solution to a specific differential equation. This method will be presented in the following section § 2. The same result can also be easily derived by applying some simple formulae from special relativity. This approach forms the basis of the second method and will be presented in section § 3.

2. Marginally trapped surfaces.

Let M be any smooth spacetime with metric $g_{\mu\nu}$ and let p^μ be a null-vector normalized such that $0 = p^\mu{}_{;\nu} p^\nu$. The condition that the divergence of p^μ vanishes is then

$$0 = p^\mu{}_{;\mu} = g^{\mu\nu} p_{\mu;\nu} \ .$$

This may also be expressed in terms of the intrinsic and extrinsic geometries of each Cauchy surface. Let S be a typical Cauchy surface in M and let $h_{\mu\nu}$ be the metric on S. The timelike unit normal to S will be denoted by n^μ and the extrinsic curvature, $h_\alpha{}^\mu h_\beta{}^\nu n_{\mu;\nu}$, by $K_{\alpha\beta}$. Now suppose that q^μ is the unit vector in S tangent to the direction of propagation of the light rays in S (ie. q^μ is the projection of p^μ onto S). The marginally trapped surface will be the closed 2-surface that is everywhere normal to q^μ in S. By writing $g^{\mu\nu} = -n^\mu n^\nu + h^{\mu\nu}$ and $p^\mu = n^\mu + q^\mu$ the zero-divergence condition may also be written as (see [2])

$$0 = -K^\alpha{}_\alpha + q^\alpha q^\beta K_{\alpha\beta} + q^\alpha{}_{|\alpha} \qquad (2.1)$$

where $q^\alpha{}_{|\alpha}$ is the covariant divergence of q^α in S.

Our aim is to obtain an appropriate solution of this differential equation when S is a simplicial space. Since the metric inside each 3-simplex of S is flat and since the extrinsic curvature is concentrated on the 2-dimensional faces of each 3-simplex (see [3]), it follows that a constant q^μ is a solution of (2.1) in each 3-simplex. The problem now is to find a suitable choice of the q^μ in each 3-simplex so that (2.1) is satisfied for any reasonable interpolation of the q^μ across the boundary between any pair of neighbouring 3-simplicies.

Let s_1 and s_2 be a pair of adjacent 3-simplicies. Their 2-dimensional interface will be denoted by s_{12}. The value of q^μ in the pair of 3-simplicies will be denoted by q_1^μ in s_1 and q_2^μ in s_2. Suppose that the 2-plane normal to q_1^μ (ie. the wave front) intersects s_{12}. The common region must be a line segment in s_{12} and will be denoted by γ_{12}. Now since a closed 2-surface is to be built, it follows that this line segment must also be the intersection of s_{12} and the 2-plane normal to q_2^μ. Thus the only quantity that varies as the wave front crosses s_{12} is the inclination of the wave front to s_{12}. The above differential equation will now be used to determine the change in inclination of the wave front in crossing s_{12}.

Let $x^\mu, \mu = 1, \cdots 3$ be a set of Euclidian coordinates in S that covers the interior of $s_1 \cup s_2$. In this frame the metric components are just $\delta_{\mu\nu}$. Choose a unit orthonormal frame u^μ, v^μ and w^μ so that v^μ and w^μ lie in s_{12} with w^μ parallel to γ_{12}. Notice that, since the metric in $s_1 \cup s_2$ is flat,

$$0 = u^\mu{}_{;\nu} = v^\mu{}_{;\nu} = w^\mu{}_{;\nu} \,.$$

The projections of q_1^μ and q_2^μ onto this frame may be written as

$$q_i^\mu = u^\mu \cos\rho_i + v^\mu \sin\rho_i \qquad i = 1, 2$$

where ρ_i is the angle between q_i^μ and u^μ. Clearly, the component of q_i^μ parallel to w_i^μ does not change upon crossing s_{12}. Thus the dependence upon this vector will be suppressed in the following analysis. Consider now a path, in S, from s_1 into s_2. Upon this path an interpolated q^μ may be defined by

$$q^\mu(l) = u^\mu \cos\rho(l) + v^\mu \sin\rho(l) \tag{2.2}$$

where l is the proper distance measured along the path. The function $\rho(l)$ is chosen to vary very rapidly over a short distance through s_{12}. It must also be chosen so that $\rho = \rho_1$ in s_1 and $\rho = \rho_2$ in s_2.

In an earlier paper [3] it was shown that, in the neighbourhood of s_{12},

$$K_{\mu\nu} = u_\mu\, u_\nu\, \frac{d\beta}{du} \tag{2.3}$$

where u is the proper distance measured along u^μ and β is the angle between the timelike unit normal (suitably interpolated across s_{12}) on S and some constant timelike vector on $s_1 \cup s_2$ (the metric is flat, thus such a vector can always be constructed). The change in β in crossing s_{12} is just the boost angle that maps the rest frame of s_1 into the rest frame of s_2. The derivative, $d\beta/du$, behaves like a Dirac delta-function on s_{12}.

A substitution of (2.2) and (2.3) into (2.1) will lead to

$$0 = -\frac{d\beta}{du}\sin^2\rho - u^\mu \rho_{,\mu} \sin\rho + v^\mu \rho_{,\mu} \cos\rho \,. \tag{2.4}$$

A simple expression for $\rho_{,\mu}$ can be obtained by noting that the metric in $s_1 \cup s_2$ is invariant with respect to translations parallel to s_{12}. Thus, by generating a family of paths from s_1 to s_2 (eg. by Lie dragging the original path along a vector parallel to s_{12}), it follows that ρ depends only upon the distance measured away from s_{12}. Consequently

$$\rho_{,\mu} = u_\mu \frac{d\rho}{du}$$

which upon substitution into (2.4) will lead to

$$0 = -\frac{d\beta}{du} \sin^2 \rho - \frac{d\rho}{du} \sin \rho \, . \tag{2.5}$$

For the moment suppose that $\sin \rho \neq 0$. The differential equation is then rather easy to integrate, with the result that

$$\Delta\beta = \Delta \left(\tanh^{-1} (\cos \rho) \right) \, . \tag{2.6}$$

The singular solution, $\sin \rho = 0$, arises when s_{12} is a piece of the marginally trapped surface.

3. The aberration formula.

Let T_1 and T_2 be the 4-dimensional timelike tubes representing the evolution of s_1 and s_2 respectively. The metric throughout $T_1 \cup T_2$ is flat. Let $c_i, i = 1, 2$ be the two pieces of the marginally trapped surface in $s_i, i = 1, 2$. Consider the family of outward pointing null geodesics to c_1. Since the metric in $T_1 \cup T_2$ is flat it follows that for any flat spatial cross section of this family the divergence of the null geodesics must also be zero. The pieces c_1 and c_2 are therefore the cross sections of this family generated by the intersection of the family with s_1 and s_2 respectively. Now consider any one point on c_1. The projection of the geodesic onto $s_1 \cup s_2$ will be a path from s_1 into s_2. This is the path of the light ray in S. If the rest frames of s_1 and s_2 are in relative motion then the appearance of this light ray must differ between s_1 and s_2.

Choose a set of Lorentzian coordinates $x^\mu, \mu = 1, \cdots 4$ throughout $T_1 \cup T_2$. Let p^μ be the components of the null vector and let ${n^\mu}_i$ be the unit timelike normal to s_i. Choose v^μ

and w^μ to be unit spatial vectors tangent to s_{12} with w^μ chosen so that $0 = p^\mu w_\mu$. Finally choose ${u^\mu}_i$ to be unit vector that completes the tetrad in T_i. The rest frames of s_1 and s_2 are related by the Lorentz transformation

$$
\begin{aligned}
{n^\mu}_2 &= {n^\mu}_1 \cosh \Delta\beta + {u^\mu}_1 \sinh \Delta\beta \,, \\
{u^\mu}_2 &= {n^\mu}_1 \sinh \Delta\beta + {u^\mu}_1 \cosh \Delta\beta \,,
\end{aligned}
$$

where $\Delta\beta$ is the boost angle. The projection of p^μ onto the two tetrads is

$$
p^\mu = \lambda_1 \left({n^\mu}_1 + u_1 \mu \cos\rho_1 + v^\mu \sin\rho_1 \right) = \lambda_2 \left({n^\mu}_2 + u_2 \mu \cos\rho_2 + v^\mu \sin\rho_2 \right)
$$

where $\lambda_i, i = 1, 2$ are a pair of constants. Combining this expression with the above Lorentz transformation and a subsequent elimination of the λ_i will lead to

$$
\cos\rho_1 = \frac{\cos\rho_2 + \tanh\Delta\beta}{1 + \cos\rho_2 \tanh\Delta\beta} \,.
$$

This is the usual aberration formula from special relativity. It is rather easy to show that this equation and equation (2.6) are equivalent.

4. Discussion.

Equation (2.6) is the principal result of this paper. If a piece of a marginally trapped surface has been constructed in one 3-simplex then that equation may be used to extend this surface to the neighbouring 3-simplicies. Whether a continued application of this construction will lead to a closed 2-surface is a matter of trial and error. It has been suggested [4] that in the construction of initial data for spaces with many black holes the marginally trapped surfaces should be built into the space as a boundary condition. This is motivated by the fact that the marginally trapped surface will always be contained within the black hole. Thus spacetimes built from this condition will contain all the information relevant to external observers. Building the space in this way also avoids the trial and error method of searching for the trapped surfaces. A systematic method of constructing marginally trapped surfaces that are guaranteed to be closed has been presented by Nakamura *et al.* [2]. Whether or not a related construction, with a similar guarantee, can be developed for the Regge calculus is an open question.

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