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THE ADM ENERGY AND 3-MOMENTUM FOR A SIMPLICIAL SPACE.

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The theory of distributions will be used in computing the ADM energy for a finite simplicial space. A related method will be presented for the computation of the ADM 3-momentum. It will be shown that the energy is proportional to the contribution to the integral curvature from the legs on the boundary of the space.

1. Introduction.

Although the Regge calculus deals entirely with geometric quantities the translation of those quantities into physically interesting quantities seems rather obscure. For example, how would one calculate the momentum or the energy of a simplicial space given only a set of leg lengths? The problem appears to arise from our limited understanding of the relationship between continuum and discrete based concepts. Thus even though there are well defined procedures for calculating the physical quantities in a continuous space those procedures can not be applied directly to discrete or simplicial spaces.

Two examples of how these difficulties may be overcome, in the computation of the ADM energy and 3-momentum [1,2], are the principal results of this paper. The basic procedure will be to re-interpret the usual integral formula as the value of a distribution on a suitable test function. This distribution will then be applied to a simplicial space. Ultimately, after some standard manipulations, a geometric expression for the ADM energy arises. This part of the work will be presented in section § 3. Unfortunately a number of technical difficulties precludes the use of this method in the calculation of the ADM 3-momentum. Thus in section § 4 an alternative method will be presented. The definitions of asymptotic flatness, the choice of coordinates and the values of the metric components for the simplicial space will be presented in the section § 2. Our analysis will be restricted to finite simplicial spaces in which each 3-simplex is a tetrahedron. The generic n-simplex will always be denoted by σ_n. The notation $\sigma_m(\sigma_n)$ will appear frequently and will represent some set of m-simplicies associated with this n-simplex. Latin indicies will cover the values 1,2 and 3.

This paper is a sequel to a recent paper by the author [3] in which a detailed use of the theory of distributions for weak simplicial spacetimes was presented. The style and notation of this paper are based upon that paper.

2. Asymptotically flat simplicial spaces.

A notion of asymptotic flatness for a simplicial space might well be based upon the rate at which the defects decay as one chooses legs closer and closer to the boundary of the space. However this approach would not yield a satisfactory definition. For example suppose it were possible to attach a flat external space to the boundary of the original space. It would then be possible to calculate the defects on those legs in the boundary of the original space. However, the fact that the defects on the neighbouring internal legs are weak does not guarantee that the defects on the boundary should also be weak. It is for this reason that the following assumptions are made.

i) The original simplicial space may be attached to a flat external space.

ii) The metric is continuous across the boundary between the two spaces.

iii) The defects on the legs on and near the boundary of the original space are all weak (ie. the defects are dominated by the leading term in the perturbation series.)

Simplicial spaces that satisfy these conditions will, in this paper, be referred to as being asymptotically flat. This definition is nowhere near as precise as that used for continuous spacetimes [4,5]. It is however sufficient for our purposes.

The original and extended spaces will be denoted by Σ and $\overline{\Sigma}$ respectively. Their respective boundaries will be denoted by S and $\overline{S}$.

The geometry of a simplicial space is normally represented by a table of leg-lengths and a table of inter-vertex connections. However the integral formula for the ADM energy requires a coordinate frame that is asymptotically Euclidian in $\overline{\Sigma}$. Thus our first task will be to establish a relationship between the leg-lengths and the coordinate frame on $\overline{\Sigma}$. Since the geometry external to Σ is flat it is always possible to choose Euclidian coordinates in $\overline{\Sigma}-\Sigma$. The metric components, h_{ij}, will then equal δ_{ij}. Now choose any reasonable extension of these coordinates into the interior of Σ. Let x^i_a be the coordinates of the vertex a in Σ. The metric components are piecewise constants in each 3-simplex of Σ and will, in general, differ from δ_{ij}. Their values may be calculated from

$$L^2_{ab} = h_{ij}\,\Delta x^i_{ab}\Delta x^j_{ab} \tag{2.1}$$

where L_{ab} is the length of the leg joining vertices a and b while $\Delta x^i_{ab} = x^i_a - x^i_b$. By this method one can tabulate the h_{ij} in each of the 3-simplicies of Σ.

It will be assumed throughout most of the next section that the simplicial space is not only asymptotically weak but that it is globally weak. Thus the $h_{ij} \approx \delta_{ij}$ in Σ and all of the terms in the integrals will be accurate to first order in $h_{ij} - \delta_{ij}$. Indicies will be raised and lowered using δ_{ij}. This assumption is not essential (and in fact will be removed at the end of the analysis) but it does make the analysis a little more transparent than would otherwise be the case.

3. The ADM energy.

If the metric components h_{ij} were smooth on Σ then the ADM energy [1,2] could be calculated as

$$E = \frac{1}{16\pi} \int_S \left(h_{ij,i} - h_{ii,j}\right)\, n_j \, d^2A \tag{3.1}$$

where n_j are the components of the unit-normal to S and d^2A is the area element on S. Our aim is to interpret this integral for simplicial spaces.

There are two problems in applying this equation directly to a simplicial space. The most obvious is that the metric components h_{ij} are not differentiable. This problem will be overcome by viewing these components as distributions. The above integral will then arise as the value of a certain distribution on a suitably chosen test function. The second and not so obvious difficulty is that the integrand involves derivatives in all three coordinates. Thus the double integral will only remove two of the three derivatives. A method of avoiding this problem is to first re-write the integral as a volume integral. This later integral will be used as the basis for the subsequent analysis.

Using Gauss's theorem the above integral may be re-written as

$$E = \frac{1}{16\pi} \int_\Sigma \left(h_{ij,ij} - h_{ii,jj}\right)\, d^3V$$

where d^3V is the volume element on Σ. Now consider the distribution

$$I(f) = \int_\Sigma \left(h_{ij,ij} - h_{ii,jj}\right)\, f(x)\, d^3V \tag{3.2}$$

where $f(x)$ is any test function with compact support on $\overline{\Sigma}$ (ie. f and all of its derivatives vanish on and outside $\overline{S}$). The ADM energy will then be the value of this distribution when $f = 1/16\pi$ throughout Σ (such a test function can always be constructed, see [6,7]).
Since f and its derivatives vanish on $\overline{S}$ it follows that after two integrations by parts one may also write

$$I(f) = \int_{\overline{\Sigma}} \left(h_{ij} f_{,ij} - h_{ii} f_{,jj}\right)\, d^3V.$$

However for a simplicial space the h_{ij} are piecewise constant in each of the simplicies of the space. Thus one also has

$$I(f) = -2\int_{\overline{\Sigma}-\Sigma} f_{,ii}\, d^3V + \sum_{\sigma_3(\Sigma)} \int_{\sigma_3} \left(h_{ij} f_{,ij} - h_{ii} f_{,jj}\right)\, d^3V$$

where the summation includes all of the 3-simplicies of Σ. The integral throughout each 3-simplex may be converted, using Gauss's theorem, to surface integrals over the faces of the 3-simplicies. This leads to

$$I(f) = -2\int_{\overline{\Sigma}-\Sigma} f_{,ii}\, d^3V + \sum_{\sigma_3(\Sigma)} \sum_{\sigma_2(\sigma_3)} \int_{\sigma_2} \left(h_{ij} f_{,i} - h_{ii} f_{,j}\right) n_j\, d^2A$$

in which the inner sum includes each of the four faces of each tetrahedron, n_j are the components of the unit normal to each face and d^2A is the element of area on each face. Notice that the n_j are constant on each face. Before applying Gauss's theorem once again it is important to notice that the integrand contains derivatives in all three coordinates. This problem can be overcome in the following way. First define A_i by

$$A_i = \left(h_{ij} n_j - h_{jj} n_i\right)\, f$$

then the integrand is just $A_{i,i}$. Now consider one face and choose (temporarily) the orthogonal coordinates x^0, x^1, x^2 so that x^0 is measured normal to the face. Let n be the proper distance measured along x^0 then

$$A_{i,i} = \frac{dA_\perp}{dn} + \sum_{j=1}^{2} A_{\parallel j,j}$$

where

$$A_\perp = A_i n_i$$

$$A_{\|j} = A_j - n_j A_\perp \, .$$

Gauss's theorem may then be applied to the term involving the $A_{\|j,j}$. The result (expressed in the original coordinates) is

$$\begin{aligned} I(f) = &- 2 \int_{\overline{\Sigma}-\Sigma} f_{,ii} \, d^3V \\ &+ \sum_{\sigma_3(\Sigma)} \sum_{\sigma_2(\sigma_3)} \int_{\sigma_2} (h_{ij} n_i n_j - h_{ii} n_j n_j) \frac{df}{dn} \, d^2A \\ &+ \sum_{\sigma_3(\Sigma)} \sum_{\sigma_2(\sigma_3)} \sum_{\sigma_1(\sigma_2)} \int_{\sigma_1} h_{ij} n_i m_j \, f \, dL \end{aligned}$$

where the sum over σ_1 includes the three legs of each face, dL is the element of length on σ_1 and m_i is the outward pointing unit normal to σ_1 and tangent to σ_2.

This expression may be further simplified by first noting that in the double sum each σ_2 on the interior of Σ is counted twice, once with one orientation for n_i and once with the opposite orientation. Thus these terms must cancel leaving only the contributions from the $\sigma_2{}'$s on the boundary S. The expression may now be written as

$$\begin{aligned} I(f) = &- 2 \int_{\overline{\Sigma}-\Sigma} f_{,ii} \, d^3V \\ &+ \sum_{\sigma_3(\partial\Sigma)} \sum_{\sigma_2(\sigma_3)} (h_{ij} n_i n_j - h_{ii} n_j n_j) \int_{\sigma_2} \frac{df}{dn} \, d^2A \\ &+ \sum_{\sigma_3(\Sigma)} \sum_{\sigma_2(\sigma_3)} \sum_{\sigma_1(\sigma_2)} h_{ij} n_i m_j \int_{\sigma_1} f \, dL \, . \end{aligned}$$

The ADM energy may now be obtained by choosing $f = 1/16\pi$ in Σ while setting f and its derivatives equal to zero on and outside S. If these conditions had been imposed in the larger region $\overline{\Sigma}$ then the ADM energy would vanish since in the region of $\overline{S}$ the 3-metric is exactly flat. Upon making this choice one has

$$E = \frac{1}{16\pi} \sum_{\sigma_3(\Sigma)} \sum_{\sigma_2(\sigma_3)} \sum_{\sigma_1(\sigma_2)} h_{ij} n_i m_j \, L \, \omega(\sigma_1, \Sigma) \; , \tag{3.3}$$

where L is the length of the leg σ_1 and $\omega(\sigma_1, \Sigma) = 1$ only when σ_1 does not lie on S, otherwise $\omega = 0$.

This expression would appear to contain contributions from all of the legs in Σ. In fact most of the terms arising from inside Σ cancel. This can be seen by looking at the simple identity

$$0 = \sum_{\sigma_1(\sigma_2)} m_i L \; . \tag{3.4}$$

This identity can be applied to all of the triangles that lie totally within Σ (ie. $\omega = 1$ for each leg) . This leaves contributions from only those triangles that have one or more legs lying on the boundary S. If all three legs of the triangle lie on S then there will be no contribution from that triangle. Thus one need only consider those triangles that have exactly one leg lying in S. Let σ_1' be one such leg and suppose that there are m tetrahedra and $m + 1$ triangles attached to this leg. Denote the ordered sequence of tetrahedra and triangles by $\sigma_3(i), i = 1, 2, \cdots m$ and $\sigma_2(i), i = 1, 2, \cdots m + 1$ respectively. Each of the interior triangles $\sigma_2(i), i = 2, 3, \cdots m$ will share a pair of tetrahedra and will therefore appear twice in the triple sum, once with one orientation for n_i and once with the opposite orientation. Thus the contribution to the triple sum for each interior triangle will be of the form $L n_i m_j \Delta h_{ij}$. However for the two triangles $\sigma_2(1)$ and $\sigma_2(m + 1)$ that lie on S there will be no such contribution. It is convenient to introduce extra terms so that there is a similar contribution from each of these triangles. From (3.4) it follows that

$$0 = \sum_{\sigma_2'(S)} \sum_{\sigma_1(\sigma_2')} L \, n_i m_j \, \Delta h_{ij}$$

where $\Delta h_{ij} = h_{ij} - \delta_{ij}$ and h_{ij} are the metric components of the tetrahedron based on the face σ_2'. The summation includes only those faces lying on S. This sum can now be added to (3.3) without altering the value of the energy. There are now sufficient terms in this expression so that for every triangle attached to each σ_1' there is a contribution of the form $Ln_i m_j \Delta h_{ij}$. One final simplification needs to be made. By once again using the identity (3.4) it is easy to show that

$$(m_i L)_{\sigma_1'} = - \sum_{\sigma_1(\sigma_2)} m_i L \, \omega(\sigma_1, \Sigma) \, .$$

Upon combining each of the above re-arrangements the result will be

$$E = \frac{-1}{16\pi} \sum_{\sigma_1(S)} \sum_{\sigma_2(\sigma_1)} L \, n_i m_j \, \Delta h_{ij}$$

where the sum over σ_1 includes only those legs on S and the sum over σ_2 includes all of the triangles attached to each such leg.

In a previous paper [3] it was shown that for a weak simplicial spacetime the defect θ is related to the metric perturbations $\gamma_{\alpha\beta}$ by

$$2\theta = \sum_{\sigma_3(\sigma_2)} n_\mu m_\nu \, \Delta\gamma_{\mu\nu} \, .$$

It is easy to see that for the space Σ the appropriate formula for the defect on the legs on and near S would be

$$2\alpha = \sum_{\sigma_2(\sigma_1)} n_i m_j \, \Delta\tau_{ij}$$

where α is the defect and τ_{ij} are the perturbations in the 3-metric (ie. $\tau_{ij} = h_{ij} - \delta_{ij}$). Using this equation the above expression may be simplified to just

$$E = \frac{-1}{8\pi} \sum_{\sigma_1(S)} \alpha \, L \, . \tag{3.5}$$

This is our final expression for the ADM energy. It was derived under the assumption that the global geometry was weak. This condition can now be relaxed since the result depends only upon the geometry near S and *not* upon the geometry deep in the interior of S. Thus this result applies to *any* asymptotically flat simplicial spacetime. It shows that the energy of the simplicial space is proportional to the contribution to the integral curvature from the bones on the boundary when the external geometry is flat.

4. The ADM 3-momentum.

The ADM 3-momentum for the space Σ may be calculated as [1,2]

$$P_i(S) = \frac{1}{8\pi}\int_S \left(K_{ij}N^j - KN_i\right)\, d^2A \tag{4.1}$$

where K_{ij} is the extrinsic curvature tensor of the space, N_i is the unit normal to the boundary, S, of the space and d^2A is the element of area on S. It is natural to ask whether the techniques just presented might also be suitable in evaluating this integral when S is a simplicial space. The most obvious difficulty in applying the above expression to a simplicial space is that the integrand is not a well behaved function. This same problem arose in the previous sections where it was shown that the integrand possessed a delta-function distribution on each leg lying in S. The integration over the 2-surface S could therefore be dealt with rather easily. For the above integral, (4.1), the situation is not so easy. The problem is that the K_{ij} in the integrand give rise to delta-functions concentrated on the triangles of S [8]. To obtain a meaningful value for the integral it will therefore be necessary to perform the integration not over the triangles but on some surface that cuts through the triangles. This will be done by introducing an artificial 2-surface S' as a slightly shrunken version of S. The ADM 3-momentum for S will then be defined as the limit of the ADM 3-momentum for S' as S' is expanded to coincide with S. Alternatively, it could be argued that since the space is asymptoticaly flat the same 3-momentum should arise irrespective of whether the calculations are performed on S or S'. The calculations will proceed as follows. The surface S' will be partitioned into a set of non-overlapping regions each containing just one leg (arising from the intersection of a triangle with S'). As the integrand is linear in K_{ij} there will not be any interactions amongst the adjacent triangles. The integral over S' will, therefore, be reduced to a sum over the separate integrals for each of the sub-regions in the partition of S'. Each term in the sum will be evaluated, terms will be grouped to obtain the cotribution from each leg after which these groups will be summed to yield the final expression.

Consider a typical leg in S and denote that leg by σ_1. Suppose that there are n triangles attached to this leg. Now let $\sigma_2(j), j = 1, 2, 3 \cdots n$ represent an ordered sequence of those triangles. The first and last triangles lie in S and thus will not intersect S'. Let $\sigma_1'(j)$ represent the leg generated by the intersection of $\sigma_2(j)$ with S' and let L_j' represent the length of that leg. Consider a partition of S' into a set of regions $S_i', i = 1, 2, 3 \cdots$ such that S_j' contains only one leg, $\sigma_1'(j)$. Now consider the integral

$$P_i(\sigma_1, S_j') = \frac{1}{8\pi} \int_{S_j'} (K_{ik} N_k - K N_i)\ d^2A \qquad j = 2, 3 \cdots n - 1\,. \tag{4.2}$$

To evaluate this integral first choose a unit orthonormal basis u^i, v^i, w^i such that u^i is normal to $\sigma_2(j)$ and v^i is normal to $\sigma_1'(j)$. It follows that w^i must be tangent to $\sigma_2(j)$ and parallel to $\sigma_1'(j)$. Now let (u, v, w) be a set of coordinates measured along the integral curves of this basis. Construct also the orthogonal coordinates (w, x, y), with x measured normal to S_j'. A pictorial representation of these symbols is presented in Figs.(1a,b). It has previously been shown [8] that, for each $\sigma_2(j)$,

$$K_{ij} = u_i u_j \frac{d\beta}{du} \tag{4.3a}$$

and

$$\Delta\beta = \int_C d\beta \tag{4.3b}$$

where C is any simple path that crosses $\sigma_2(j)$ and $\Delta\beta$ is the boost parameter for the Lorentz transformation between the rest frames of the blocks on either side of $\sigma_2(j)$. Upon using $(4.3a)$ and

$$u_i N_i = \cos\rho\,,$$

$$du = dy\ \sin\rho\,,$$

$$N_i = u_i\ \cos\rho - v_i\ \sin\rho\,,$$

where ρ is the angle between $\sigma_2(j)$ and S'_j, the previous integral may be reduced to

$$P_i(\sigma_1, S'_j) = \frac{1}{8\pi} \int_{S'_j} v_i \, \frac{d\beta}{dy} \, d^2 A \, .$$

However the element of area d^2A on S'_j is just $dwdy$, thus the integration is easily performed with the result that

$$P_i(\sigma_1, S'_j) = \frac{1}{8\pi} v_i \, L'_j \, \Delta\beta \, . \tag{4.4}$$

This expression is, clearly, well defined in the limit as $S' \to S$. There is one such expression for each $\sigma_2(j), j = 2, 3 \cdots n-1$. Thus the nett contribution to the ADM 3-momentum from this single leg, σ_1, in S is given by

$$P_i(\sigma_1) = \frac{L}{8\pi} \sum_{j=2}^{n-1} v_i \, \Delta\beta$$

where L is the length of σ_1 (note that $L'_j \to L$ as $S' \to S$). In calculating this sum it will be necessary to parallel transport each vector to some predefined point in $\overline{\Sigma}$. The choice of this point and the paths along which the vectors are parallel transported will affect the final values for $P_i(\sigma_1)$. However, since the metric is assumed to be weak in this region, the various possible values for $P_i(\sigma_1)$ must differ by an order of magnitude less than $P_i(\sigma_1)$. This can be seen by noting that the v_i when parallel transported around σ_1 will incur a change of order $\Delta\beta$ thus leading to a perturbation in $P_i(\sigma_1)$ of order $(\Delta\beta)^2$. Suppose now that the point at which the above sum is evaluated is chosen in $\overline{\Sigma} - \Sigma$. Since the metric in this region is chosen to be flat it is now possible to add together the separate contributions to the ADM 3-momentum from each leg in S. This leads to

$$P_i(S) = \frac{1}{8\pi} \sum_{\sigma_1(S)} L \sum_{j=2}^{n-1} v_i \, \Delta\beta \tag{4.5}$$

as our final expression for the ADM 3-momentum for a simplicial space.

4. Discussion.

It seems natural to ask whether similar techniques might be useful in calculating certain other quantities, for example the Komar mass or the angular momentum of the space. Work is currently under progress on both of these problems. The main difficulty with the calculation of the Komar mass is that one must obtain a suitable definition, that can be expressed in terms of the leg-lengths and defects, of a timelike Killing vector in the far flat regions of the Regge spacetime. The nature of the symmetry groups and gauge transformations in the Regge calculus has been a problem for some time. If one is to use such symmetries to compute certain physical quantities then this problem must be resolved.

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Figures.

Fig. 1a. A perpendicular cross-section through one of the legs σ_1 in S. The dashed lines indicate a continuation of the surfaces.

Fig. 1b. A perspective view of one of the triangles $\sigma_2(j)$ and its intersection with a portion of the surface S'. Notice that the sub-region S'_j may be of any shape provided it encloses the leg $\sigma'_1(j)$.

Fig. 1c. As for Fig. 1b. but viewed along the leg $\sigma'_1(j)$.