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EQUIVALENCE OF THE REGGE AND EINSTEIN EQUATIONS FOR ALMOST FLAT SIMPLICIAL SPACETIMES.

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The theory of distributions is applied to almost flat simplicial spacetimes. Explicit expressions are given for the first order defects. It is shown explicitly that the Riemann tensor for an almost flat simplicial spacetime contains delta-functions on the bones and derivatives of delta-functions on the 3-dimensional faces of the boundary of the spacetime. These later terms have not previously been seen in the Regge calculus. It is shown that the Regge and Hilbert actions have equal values on almost flat simplicial spacetimes and that the Einstein equations lead directly to the Regge field equations.

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1. Introduction.

It is often stated that the Regge calculus is an appropriate generalization of Einstein's theory of General Relativity to simplicial spacetimes. The argument goes that the Riemann tensor behaves like a sequence of delta-functions based on the bones of the simplicial spacetime and so certain integrals (in particular the Hilbert action integral) may be readily discretized. If there happened to be a subtle error in this popular heuristic argument then the value of the Regge calculus as an approximation to General Relativity must be questioned. Thus it is of paramount importance to try to understand the relationship between the two theories. Unfortunately this task is fraught with many mathematical difficulties. The main difficulty arises from the non-linear structure of the Riemann tensor as a function of the metric. Thus if one attempts to construct the metric as a distribution (rather than as a point function) the formal calculation of the Riemann tensor will lead to non-linear combinations of various delta-functions. Generally speaking such combinations are not acceptable in the standard formulation of generalized functions. One may well be asking too much – it may be wise to tackle an easier problem. Thus in this paper it will be assumed that the spacetime is sufficiently weak that all non-linear terms may be discarded. If the two theories are to agree then surely they must agree when the spacetimes are weak.

This idea, to compare the theories for weak spacetimes, has been investigated by others [1,2,3,4,5] . The work of Roček and Williams [1,2] and the later work of Williams [3] was intended to be a prelude to their attempts to form a quantum theory of the Regge calculus. Their calculations were couched in the terminology of quantum field theory and were tied to a specific choice of simplicial spacetime (a hyper-cubic lattice). Their method of calculation was to perform a Fourier analysis on the linearized field equations. They then showed that when the discretization size was sufficiently small the resulting spectrums were identical in the long wavelength limit. The principal impediment in applying this method to other simplicial spacetimes is the shear bulk of the calculations.

The central assumptions in the work of Friedberg and Lee [4] and also in Feinberg *et al* [5] is that any smooth space may be arbitrarily approximated by a simplicial space and, furthermore, that the simplicial space may be represented by a sequence of smooth geometries. In examining the field equations for weak spacetimes they chose to look, primarily, at the graviton propagator. However they did not present an explicit relation for the linearized defects in terms of the perturbed leg-lengths.

A somewhat different approach was adopted by Cheeger, Müller and Schrader [6]. They showed that if a given sequence of discrete geometries converged to a well defined smooth geometry then the value of the Regge action converged to that of the Hilbert action. This result is true in the full non-linear theory. In our approach the question will be to ask whether or not the Regge and Hilbert actions, when evaluated on one discrete geometry, are equal.

The methods to be presented here differs from those of the above authors by employing the theory of distributions in all calculations. By this method it will be shown that all of the above assertions (ie. that the Riemann tensor is a sequence of delta-functions, that the Einstein and Regge equations are equivalent) are valid. In the process one obtains certain boundary terms. These terms may well be important in the analysis of the asymptotic behavior of non-weak but asymptoticaly flat simplicial spacetimes.

In the following two sections a representation of the metric for weak spacetimes will be presented. In section § 4 the relevant aspects of the theory of distributions will be introduced and applied to the calculation of the (distributions associated with the) Riemann scalar and Ricci tensor. In the final section § 5 an equivalence between the Einstein and Regge equations for weak simplicial spacetimes will be established. The occurrence of null-bones in a generic simplicial spacetime is unlikely and is therefore rather exceptional. Thus it will be assumed throughout this paper that there are no null-bones in the simplicial spacetimes. The basic 4-dimensional building blocks of the simplicial spacetimes will be assumed to be 4-simplicies. The generic n-simplex will always be denoted by σ_n while a particular n-simplex will be denoted by $\sigma_n(i)$ for some specific index i.

2. Flat simplicial spacetimes.

Globally flat simplicial spacetimes are rather easy to generate. An invertible map of the vertices into some portion of Minkowski space induces a flat metric throughout the simplicial spacetime. Let the various vertices of the simplicial manifold be represented by $\sigma_0(i), i = 1, 2, 3\cdots$. For each vertex $\sigma_0(i)$ the associated coordinates in the Minkowski space will be denoted by $x^\mu(i)$. The components of the metric are just the usual $\eta_{\mu\nu}$. The leg lengths L_{ij} are then computed as

$$L_{ij}^2 = \eta_{\mu\nu}\left\{x^\mu(i) - x^\mu(j)\right\}\left\{x^\nu(i) - x^\nu(j)\right\} \ . \tag{2.1}$$

With this choice of leg lengths all of the defects vanish. This flat simplicial spacetime will be denoted by M.

3. Almost flat simplicial spacetimes.

Small perturbations in the simplicial spacetime can be generated directly by perturbing the L_{ij} or indirectly by making small perturbations to the metric $\eta_{\mu\nu}$ and to then re-calculate the (perturbed) L_{ij}. In this later approach the original coordinates of the vertices are retained and a piecewise constant perturbation $\gamma_{\mu\nu}$ is introduced in each 4-simplex of the manifold. The resulting non-flat spacetime will be denoted by M'. Let the perturbation in L_{ij} be represented by δL_{ij}. The δL_{ij} and the $\gamma_{\mu\nu}$ must be chosen so that

$$\left(L_{ij} + \delta L_{ij}\right)^2 = \left(\eta_{\mu\nu} + \gamma_{\mu\nu}\right)\left\{x^\mu(i) - x^\mu(j)\right\}\left\{x^\nu(i) - x^\nu(j)\right\} \ . \tag{3.1}$$

It is not too hard to see that, given any set of $\delta L_{ij}{}'$s, it is always possible to solve, uniquely, for the $\gamma_{\mu\nu}$ in each 4-simplex of M'.

In the following section it will be convenient to view M' as being a proper subset of some other spacetime $\overline{M}$ in which the metric external to M' is exactly flat (ie. a flat metric is attached to the exterior of M'). This construction seems plausible since the un-perturbed simplicial spacetime M was constructed as a subset of flat Lorentzian space and thus small perturbations within M should not alter the geometry external to M.

Consider any one bone σ'_2 in M' and the set of 4-simplicies $\sigma'_4(i)$ attached to this bone. Their unperturbed counterparts in M are σ_2 and $\sigma_4(i)$. As the metric in each of the $\sigma'_4(i)$ is flat it is possible to construct an isometric map of each $\sigma'_4(i)$ into some other 4-simplex, denoted by $\sigma''_4(i)$, in M. There is considerable arbitrariness in this construction. One could choose the map so that each $\sigma''_4(i)$ is disconnected from all other $\sigma''_4(i)$. This is, of course, an extreme example. It is not too hard to see that one can always choose the map so that the image of σ'_2, denoted by σ''_2, is unique (ie. the set of $\sigma'_4(i)$ remain attached to the bone after the map). One can then re-orient the σ''_2 and the $\sigma''_4(i)$ so that σ''_2 lies in the same plane as σ_2. This is a particularly useful choice for the map since the subsequent analysis (sections § 4,5) reduces to a local analysis for each bone. Thus this map will be used in all subsequent calculations.

Consider now the two 4-simplicies $\sigma_4(1)$ and $\sigma''_4(1)$. The map carries the coordinates of the five vertices of $\sigma'_4(1)$ onto the five vertices of $\sigma''_4(1)$. These are, however, just the original coordinates on $\sigma_4(1)$. Thus one may view the map as establishing a passive transformation from $\sigma'_4(1)$ to $\sigma''_4(1)$ or as an active transformation of $\sigma_4(1)$ into $\sigma''_4(1)$. This later viewpoint will be used to develop some useful facts about the $\gamma_{\mu\nu}$.

Let the coordinates of the vertices of $\sigma_4(1)$ and $\sigma''_4(1)$ be $x^\mu(i), i = 1, \cdots 5$ and $x''^\mu(i), i = 1, \cdots 5$ respectively. The active transformation may written as

$$x''^\mu(i) = \Lambda^\mu{}_\nu x^\nu(i) \qquad i = 1, \cdots 5 \tag{3.2a}$$

for some choice of constants $\Lambda^\mu{}_\nu$. The passive transformation is then expressed as

$$\eta_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \left(\eta_{\alpha\beta} + \gamma_{\alpha\beta}\right) \ . \tag{3.2b}$$

Since the perturbations are supposed to be small it follows that

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu \tag{3.3}$$

for some set of small constants $\epsilon^\mu{}_\nu$. Suppose that p^μ, q^μ, n^μ and m^μ form a unit orthonormal tetrad in $\sigma_4(1)$ chosen so that p^μ and q^μ span the bone. The $\epsilon^\mu{}_\nu$ may be projected onto this tetrad. Any parallel-perpendicular component (ie. $\epsilon^\mu{}_\nu n_\mu p^\nu$) must vanish since the action of $\Lambda^\mu{}_\nu$ on any vector parallel to the bone must yield another vector parallel to the bone. Thus the most general form for $\epsilon^\mu{}_\nu$ will be

$$\epsilon^\mu{}_\nu = A^\mu{}_\nu(n^\alpha, m^\alpha) + B^\mu{}_\nu(p^\alpha, q^\alpha) \tag{3.4}$$

where $A^\mu{}_\nu$ and $B^\mu{}_\nu$ represent simple bilinear combinations in their arguments.

Now consider the two 3-dimensional faces (3-simplicies), attached to the bone, of $\sigma_4(1)$. Denote these faces by $\sigma_3(1)$ and $\sigma_3(2)$. Let the corresponding faces in $\sigma''_4(1)$ be $\sigma''_3(1)$ and $\sigma''_3(2)$. Suppose that m^μ_1 and n^μ_1 are a pair of unit orthogonal vectors chosen so that each is orthogonal to the bone, that m^μ_1 is parallel to $\sigma_3(1)$ and that n^μ_1 is the outward unit normal to $\sigma_3(1)$. Suppose for the moment that the bone is timelike. Then the metric in the 2-plane spanned by n^μ and m^μ will be Euclidian. Similar arguments to that which are about to be presented may be applied when the bone is spacelike. The exceptional case occurring when the bone is null will not be considered here.

Since the transformation must map $\sigma_3(1)$ onto $\sigma''_3(1)$ its action on m^μ_1 must represent a rotation of m^μ_1 in the plane spanned by m^μ_1 and n^μ_1 and also a dilation along m^μ_1 (see fig.(1)). A similar analysis applies to the action on m^μ_2. Thus one obtains

$$\Lambda^\mu{}_\nu m^\nu_1 = (1+\Delta_1) m^\mu_1 + \delta\phi_1\, n^\mu_1$$
$$\Lambda^\mu{}_\nu m^\nu_2 = (1+\Delta_2) m^\mu_2 + \delta\phi_2\, n^\mu_2$$

where $\delta\phi_1$ is the angle from $\sigma_3(1)$ to $\sigma''_3(1)$, $\delta\phi_2$ is the angle from $\sigma_3(2)$ to $\sigma''_3(2)$ and Δ_1, Δ_2 represent the (fractional) dilations. To deduce the action on n^μ_1 and n^μ_2 first write

$$n^\mu_1 = \alpha m^\mu_1 + \beta m^\mu_2$$
$$n^\mu_2 = \alpha' m^\mu_1 + \beta' m^\mu_2$$

where α, α', β and β' are to be computed from the normalization conditions. This leads to

$$\alpha = \beta' = \cot\phi\,,$$
$$\alpha' = \beta = -\operatorname{cosec}\phi$$

where ϕ is the angle between $\sigma_3(1)$ and $\sigma_3(2)$. Using these formulae one can easily verify that

$$2\delta\phi = \epsilon^{\mu}{}_{\nu}\left(n_1^{\nu} m_{1\mu} + m_1^{\nu} n_{1\mu} + n_2^{\nu} m_{2\mu} + m_2^{\nu} n_{2\mu}\right)$$

where $\delta\phi = \delta\phi_1 + \delta\phi_2$ is the total increment in the angle ϕ. Now expand (3.2b) in powers of $\epsilon^{\mu}{}_{\nu}$ and discard any terms higher than first order. This leads to

$$\gamma_{\mu\nu} = -\eta_{\alpha\nu}\epsilon^{\alpha}{}_{\mu} - \eta_{\mu\alpha}\epsilon^{\alpha}{}_{\nu}$$

and consequently

$$2\delta\phi = -\gamma_{\mu\nu}(n_1^{\mu} m_1^{\nu} + n_2^{\mu} m_2^{\nu})\ .$$

Let $\Delta\theta$ be the defect on the bone. Then

$$\begin{aligned}\Delta\theta &= -\sum_{\sigma_4(\sigma_2)} \delta\phi \\ &= \frac{1}{2}\sum_{\sigma_4(\sigma_2)} \gamma_{\mu\nu}(n_1^{\mu} m_1^{\nu} + n_2^{\mu} m_2^{\nu})\end{aligned}$$

in which the summation includes each of the $\sigma_4(i)$ attached to the bone. This can be simplified by noting that to each n_2^{μ}, m_2^{μ} in one σ_4 there is another, oppositely oriented, n_1^{μ}, m_1^{μ} in a neighbouring σ_4. Thus one obtains the important formula, accurate to first order,

$$2\Delta\theta = \sum_{\sigma_3(\sigma_2)} n^{\mu} m^{\nu} \Delta\gamma_{\mu\nu} \tag{3.5}$$

for the defect $\Delta\theta$ on the bone σ_2. Only those interfaces (ie. 3-simplicies) attached to this bone are included in the sum. The $\Delta\gamma_{\mu\nu}$ represents the change in $\gamma_{\mu\nu}$ across the interface. Similar arguments may be used to show that this expression is also valid when the bone is spacelike. This same formula will be obtained in the following section by a completely different procedure.

One final point needs to be made. The simple form for $\epsilon^{\mu}{}_{\nu}$ (3.4) when substituted into the above linear expansion for $\gamma_{\mu\nu}$ will lead to

$$\gamma_{\mu\nu} = An_\mu n_\nu + B(n_\mu m_\nu + n_\nu m_\mu) + Cm_\mu m_\nu + Dp_\mu p_\nu + E(p_\mu q_\nu + p_\nu q_\mu) + Fq_\mu q_\nu \qquad (3.6)$$

for some set of constants $A, \cdots F$. Notice that D, E and F depend only on the choice of bone and not on the choice of σ_4 (ie. they have the same values for each of the $\gamma_{\mu\nu}$'s associated with this bone). To prove this recall that since p^μ and q^μ are parallel to the bone thus $p^\mu_{;\nu} = q^\mu_{;\nu} = 0$ and consequently $0 = (\gamma_{\mu\nu}p^\mu q^\nu)_{;\alpha} = E_{,\alpha}$. Thus $0 = \Delta D = \Delta E = \Delta F$ across any σ_3. This result will be used in simplifying one of the results (4.4) of the following section.

4. Linearized Regge calculus.

The metric components $g_{\mu\nu}$ have been constructed as piecewise constants in each 4-simplex of the spacetime M. One must therefore expect delta functions, and possibly their derivatives, to arise from the action of a linear differential operator on the metric. There are two standard ways of handling this situation [7,8]. In both approaches an integration over some region of the spacetime is used to effect an inversion of the differential operator. In this way one can obtain finite expressions. Suppose that D is some linear differential operator acting on the piecewise constant function A. In one approach the function A is approximated by a sequence of smooth differentiable functions $\overline{A}_\lambda$. The sequence is chosen so that $\lim_{\lambda\to 0} \overline{A}_\lambda = A$ almost everywhere (ie. except possibly on the points where A is discontinuous). One then attempts to fully evaluate the integral

$$I = \int_{\overline{M}} D(\overline{A})\, d^4V$$

as a function of λ and where $\overline{M}$ is some region of the spacetime. The action of D on A is then defined as $\lim_{\lambda\to 0} I$ (provided the limit exists). This approach will not be used here. In the alternative approach one begins by choosing any appropriate test function. Such functions, together with all of their derivatives, must be bounded, infinitely differentiable and have compact support on $\overline{M}$. One then constructs the integral

$$I = \int_{\overline{M}} D(A) f\, d^4V$$

and attempts to re-write this as

$$I = \int_{\overline{M}} AD^*(f)\, d^4V$$

for some linear operator D^*. The integrand is now well behaved and so the region of integration may reduced to a sum over the 4-simplicies, in each of which A is constant, of $\overline{M}$. Thus one obtains

$$I = \sum_{\sigma_4(\overline{M})} A(\sigma_4) \int_{\sigma_4} D^*(f)\, d^4V$$

where the summation includes all of the 4-simplicies inside $\overline{M}$. The characterization of $D(A)$ (ie. the terms with delta functions) can then be inferred from the integral of D^* on f. The operator D^* is usually found be an integration by parts with any boundary terms being discarded (since f and its derivatives vanish on the boundary). This is the approach that will be used throughout this paper.

These ideas will now be applied to the distributions associated with the scalar curvature and the Ricci tensor. Thus consider the two integrals

$$I_1(f) = \int_{\overline{M}} Rf\sqrt{-g}\, d^4V\ , \tag{4.1a}$$

$$I_2(f^{\mu\nu}) = \int_{\overline{M}} R_{\mu\nu} f^{\mu\nu}\sqrt{-g}\, d^4V \tag{4.2a}$$

where f and $f^{\mu\nu}$ are test functions on $\overline{M}$. By writing

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}$$

the above integrals may be written, accurate to first order in $\gamma_{\mu\nu}$ [9,10], as

$$I_1(f) = \int_{\overline{M}} \left(\gamma_{\alpha\beta}{}^{,\alpha\beta} - \gamma^{,\alpha}{}_{,\alpha}\right) f\, d^4V\ , \tag{4.1b}$$

$$I_2(f^{\mu\nu}) = \frac{1}{2}\int_{\overline{M}} \left(\gamma^{\alpha}{}_{\nu,\mu\alpha} + \gamma^{\alpha}{}_{\mu,\nu\alpha} - \gamma_{\mu\nu,\alpha}{}^{,\alpha} - \gamma^{\alpha}{}_{\alpha,\mu\nu}\right) f^{\mu\nu}\, d^4V\ . \tag{4.2b}$$

Notice that, once again, indicies are raised and lowered using $\eta_{\mu\nu}$. Consider, for the moment, the first of these integrals. By applying an integration by parts twice the integral may also be written as

$$I_1(f) = \int_{\overline{M}} \left(\gamma_{\alpha\beta} f^{,\alpha\beta} - \gamma f_{,\alpha}{}^{,\alpha} \right) \, d^4V$$

where $\gamma = \gamma^\alpha{}_\alpha$. However $\gamma_{\mu\nu}$ is piecewise constant in each of the σ_4's in $\overline{M}$. They also vanish in $\overline{M} - M$. This leads to

$$I_1(f) = \sum_{\sigma_4(M)} \int_{\sigma_4} H^{\mu\nu} f_{,\mu\nu} \, d^4V$$

where

$$H^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}\gamma_{\alpha\beta} - \eta^{\mu\nu}\gamma \, .$$

The $H^{\mu\nu}$ are constant in each of the σ_4's. The integral over each 4-simplex can now be simplified by using Gauss's theorem. Let n^μ be the unit normal (ie. $n^\alpha n_\alpha = \pm 1$) to a typical face (a σ_3) of the 4-simplex. Then

$$\begin{aligned} I_1(f) &= \sum_{\sigma_4(M)} \int_{\sigma_4} H^{\mu\nu} f_{,\mu\nu} \, d^4V \\ &= \sum_{\sigma_4(M)} \int_{\sigma_4} \left(H^{\mu\nu} f_{,\nu} \right)_{,\mu} \, d^4V \\ &= \sum_{\sigma_4(M)} \sum_{\sigma_3(\sigma_4)} \int_{\sigma_3} H^{\mu\nu} n_\mu f_{,\nu} \, d^3V \end{aligned}$$

where d^3V is the natural volume element on σ_3 and the innermost summation includes only the five faces of σ_4. Now $H^{\mu\nu} n_\mu$ is constant on each σ_3 and so it seems appropriate to apply Gauss's theorem once again. However $(H^{\mu\nu} n_\mu f)_{,\nu}$ is not the divergence of a vector on σ_3 since it contains derivatives in all four coordinates. This problem can be resolved by choosing a set of coordinates three of which are chosen to lie in σ_3 with the remaining coordinate measured along the geodesic normal to σ_3. Then

$$(H^{\mu\nu} n_\mu f)_{,\nu} = \frac{d}{dn}(H_\perp f) + \left(H^\nu_\parallel f\right)_{|\nu}$$

where

$$H_\perp = H^{\mu\nu} n_\mu n_\nu$$
$$H^\nu_\parallel = H^{\mu\nu} n_\mu - (n^\alpha n_\alpha) n^\nu H_\perp$$

and d/dn is the derivative measured along the normal to σ_3 and the notation $(...)_{|\nu}$ represents partial differentiation in σ_3. Substituting this into the previous integral leads to

$$I_1(f) = \sum_{\sigma_4(M)} \sum_{\sigma_3(\sigma_4)} \int_{\sigma_3} \left\{ H_\perp \frac{df}{dn} + (H^\nu_\parallel f)_{|\nu} \right\} d^3V \; .$$

The integral of the second term may now be simplified using Gauss's theorem. Let σ_2 be a typical face of σ_3 and let m^μ be the unit vector in σ_3 and normal to σ_2. Then

$$\begin{aligned} I_1(f) = & \sum_{\sigma_4(M)} \sum_{\sigma_3(\sigma_4)} \int_{\sigma_3} H_\perp \frac{df}{dn} \, d^3V \\ & + \sum_{\sigma_4(M)} \sum_{\sigma_3(\sigma_4)} \sum_{\sigma_2(\sigma_3)} \int_{\sigma_2} H^\mu_\parallel m_\mu f \, d^2V \end{aligned}$$

where d^2V is the natural volume element on σ_2 and the summation over σ_2 includes only the four faces of σ_3. In the first double sum each σ_3 on the interior of M is counted twice. The unit normals n^μ in each instance are anti-parallel. Thus the interior terms cancel leaving only the terms arising from the $\sigma_3{}'$s on the boundary of M. This result together with a substitution for $H_\perp$ and $H^\mu_\parallel$ and a removal of constant factors from the integrals will lead to

$$\begin{aligned} I_1(f) = & \sum_{\sigma_3(\partial M)} \left(\gamma_{\alpha\beta} n^\alpha n^\beta - \gamma n^\alpha n_\alpha \right) \int_{\sigma_3} \frac{df}{dn} \, d^3V \\ & + \sum_{\sigma_4(M)} \sum_{\sigma_3(\sigma_4)} \sum_{\sigma_2(\sigma_3)} \gamma_{\alpha\beta} n^\alpha m^\beta \int_{\sigma_2} f \, d^2V \; . \end{aligned}$$

In the triple sum each σ_2 will appear many times. Our aim is to re-write this triple sum so as to gather together all of the contributions for each σ_2. Consider some typical σ_2 and choose any one of the σ_3's attached to this σ_2. This combination of σ_2 and σ_3 will appear twice in the summation, once with one orientation for n^μ and m^μ and once with the opposite orientation. Thus the nett contribution to this σ_2 from this σ_3 will be $\Delta\gamma_{\alpha\beta}n^\alpha m^\beta$. It is not hard to see that the expression for $I_1(f)$ may now be written as

$$\begin{aligned} I_1(f) = & \sum_{\sigma_3(\partial M)} \left(\gamma_{\alpha\beta}n^\alpha n^\beta - \gamma n^\alpha n_\alpha\right) \int_{\sigma_3} \frac{df}{dn}\, d^3V \\ & + \sum_{\sigma_2(M)} \sum_{\sigma_3(\sigma_2)} \Delta\gamma_{\alpha\beta}n^\alpha m^\beta \int_{\sigma_2} f\, d^2V \end{aligned}$$

where the summation over $\sigma_2(M)$ includes each σ_2 of M while the summation over $\sigma_3(\sigma_2)$ includes only those σ_3's attached to this σ_2. Using (3.5) this result may be simplified to

$$\begin{aligned} I_1(f) = & \sum_{\sigma_3(\partial M)} \left(\gamma_{\alpha\beta}n^\alpha n^\beta - \gamma n^\alpha n_\alpha\right) \int_{\sigma_3} \frac{df}{dn}\, d^3V \\ & + \sum_{\sigma_2(M)} 2\Delta\theta \int_{\sigma_2} f\, d^2V \end{aligned} \tag{4.3}$$

This expression displays clearly the character of the (first order) Riemann scalar density. The integrals of f over the σ_2's are representative of delta functions on the bones inside M. The strength of the delta function is twice the defect on the bone – a result that is well known [11]. There are also integrals of df/dn over the faces on the boundary of M. Such integrals represent the derivatives of delta functions. These terms have not previously been seen in the context of the Regge calculus. A geometric interpretation of these terms can be found in appendix A.

A similar procedure may be used in the evaluation of the second integral (4.2b). The main difference is in the way the various terms in the integrand are re-written as divergences. For example,

$$\gamma^\alpha{}_\mu f^{\mu\nu}{}_{,\alpha\nu} = \frac{1}{2}\left((\gamma^\alpha{}_\mu f^{\mu\nu}{}_{,\alpha})_{,\nu} + (\gamma^\alpha{}_\mu f^{\mu\nu}{}_{,\nu})_{,\alpha} \right) \ .$$

The two integration by parts are then applied, the boundary terms are isolated and the penultimate result is simplified using (3.6). The final result is

$$I_2(f^{\mu\nu}) = \frac{1}{2}\sum_{\sigma_3(\partial M)} \left(\gamma^\alpha{}_\mu n_\alpha n_\nu + \gamma^\alpha{}_\nu n_\alpha n_\mu - \gamma n_\mu n_\nu - \gamma_{\mu\nu}\right)\int_{\sigma_3} \frac{df^{\mu\nu}}{dn}\, d^3V$$

$$+\frac{1}{2}\sum_{\sigma_2(M)}\sum_{\sigma_3(\sigma_2)} \Delta\gamma_{\alpha\beta} n^\alpha m^\beta \left(n_\mu n_\nu + m_\mu m_\nu\right)\int_{\sigma_2} f^{\mu\nu}\, d^2V\ .$$

No further analysis will be conducted on any of the boundary terms and so all such terms will subsequently be written as just $B.T.'s$.

The combination $n_\mu n_\nu + m_\mu m_\nu$ are the components of the 2-metric in the plane spanned by n^μ and m^μ. Since this plane is orthogonal to the bone it follows that these components have the same values in each of the 4-simplicies attached to the bone. Thus these quantities may be brought outside of the sum over the σ_3's with the result

$$I_2(f^{\mu\nu}) = B.T.'s + \frac{1}{2}\sum_{\sigma_2(M)} \left(n_\mu n_\nu + m_\mu m_\nu\right)\sum_{\sigma_3(\sigma_2)} \Delta\gamma_{\alpha\beta} n^\alpha m^\beta \int_{\sigma_2} f^{\mu\nu}\, d^2V\ .$$

This may be further simplified using (3.5), leading to

$$I_2(f^{\mu\nu}) = B.T.'s + \sum_{\sigma_2(M)} \left(n_\mu n_\nu + m_\mu m_\nu\right)\Delta\theta \int_{\sigma_2} f^{\mu\nu}\, d^2V\ . \qquad (4.4)$$

This result is not (with the exception of the boundary terms) un-expected. One knows that for any 2-metric the metric, Ricci tensor and Riemann scalar curvature are related by $R_{\mu\nu} = g_{\mu\nu}R/2$. This fact may be used here since the 4-metric for any bone may be written as the direct product of two 2-metrics (for the 2-planes parallel and perpendicular to the bone).

The distribution associated with the (first order) Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R$, can now be constructed from the above expressions for I_1 and I_2. Begin by writing

$$\eta_{\mu\nu} = p_\mu p_\nu + q_\mu q_\nu + n_\mu n_\nu + m_\mu m_\nu$$

for some unit orthonormal tetrad with p^μ and q^μ chosen to span the bone and use this to simplify

$$I_3(f^{\mu\nu}) = \int_{\overline{M}} G_{\mu\nu} f^{\mu\nu}\, d^4V$$

$$= I_2(f^{\mu\nu}) - \frac{1}{2} I_1(\eta_{\mu\nu} f^{\mu\nu})\,.$$

The result is

$$I_3(f^{\mu\nu}) = B.T.'s - \sum_{\sigma_2(M)} (p_\mu p_\nu + q_\mu q_\nu)\, \Delta\theta \int_{\sigma_2} f^{\mu\nu}\, d^2V\,. \tag{4.5}$$

However $p_\mu p_\nu + q_\mu q_\nu$ is just the 2-metric on the bone. The components will be denoted by $b_{\mu\nu}$ and may be calculated from the simple formula [12,13]

$$b_{\mu\nu} = \frac{1}{A} \sum_{\sigma_1(\sigma_2)} \frac{1}{L} \frac{\partial A}{\partial L} \Delta x_\mu \Delta x_\nu$$

where A is the area of the bone, L is the length of an edge of the bone, Δx^μ is the vector joining the end points of that edge and the sum includes all three edges of the bone. A proof of this relation can be found in appendix B. Now for any test function $f^{\mu\nu}$ define $\overline{f}^{\mu\nu}(\sigma_2)$ and $\overline{f}(\sigma_2, \sigma_1)$ by

$$\overline{f}^{\mu\nu}(\sigma_2) = \frac{1}{A} \int_{\sigma_2} f^{\mu\nu}\, d^2V \tag{4.6a}$$

and

$$\overline{f}(\sigma_2, \sigma_1) = (\Delta x_\mu \Delta x_\nu)_{\sigma_1} \overline{f}^{\mu\nu}(\sigma_2)\,. \tag{4.6b}$$

Then

$$I_3(f^{\mu\nu}) = B.T's - \sum_{\sigma_2(M)} \sum_{\sigma_1(\sigma_2)} \frac{\Delta\theta}{L} \frac{\partial A}{\partial L} \overline{f}(\sigma_2, \sigma_1)\,. \tag{4.7}$$

5. The Regge and Einstein field equations.

The Einstein equations can be obtained by extremizing the Hilbert action integral. Thus one starts with

$$I = \int R\sqrt{-g}\, d^4x \tag{5.1a}$$

and performs small variations in the $g_{\mu\nu}$ to obtain

$$\delta I = B.T.'s + \int G_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}\, d^4x\ . \tag{5.1b}$$

In the Regge calculus one starts with the action sum

$$I' = 2\sum_{\sigma_2(M)} \Delta\theta\, A \tag{5.2a}$$

and takes variations in the leg-lengths to obtain

$$\delta I' = B.T.'s + 2\sum_{\sigma_1(M)}\ \sum_{\sigma_2(\sigma_1)} \Delta\theta\, \frac{\partial A}{\partial L}\, \delta L \tag{5.2b}$$

where the outer sum includes all legs inside M and the inner sum includes all bones attached to a particular leg. Using the results of the previous sections it is now possible to understand the relationship between this pair of approaches. The basic idea is to evaluate the above pair of integrals (5.1a,b) only on the simplicial spacetimes M and to then show that their values are equal to the pair of sums (5.2a,b) for an appropriate choice of test functions.

It is clear that the value of the Hilbert action I is just the value of the distribution $I_1(f)$ when the test function is chosen to be the unit function throughout M. (One should *not* set $f = 1$ throughout $\overline{M}$ since the integral (4.1b) may be converted to a surface integral and thus its value would be zero since all of the γ's vanish on that surface.) Upon setting $f = 1$ throughout M (and thus f and its derivatives must vanish on the boundary of M) it follows from (4.3) that

$$I = I_1(1) = I'\ . \tag{5.3}$$

This shows that, for weak simplicial spacetimes, the Hilbert and Regge actions are equal. This is exactly what one would expect. The comparison of δI with $\delta I'$ is, however, not so easy to achieve. Since the Hilbert action integral is to be evaluated only for simplicial spacetimes the variations $\delta g^{\mu\nu}$ must be chosen to represent well defined variations in the leg lengths of M. Choose any one leg, σ_1, and let $S(\sigma_1)$ be the interior of the set of 4-simplicies attached to this leg. Each 4-simplex in $S(\sigma_1)$ will be denoted by $\sigma_4(i)$ for some specific index i. It is shown in appendix C that arbitrary variations in the length of this leg can be generated only when $\delta g^{\mu\nu}$ is an appropriately chosen set of constants in each of the $\sigma_4(i)$ of $S(\sigma_1)$. The general form for $\delta g_{\mu\nu}(i)$ in $\sigma_4(i)$ is shown to be

$$\delta g_{\mu\nu}(i) = (\Delta x_\mu \Delta x_\nu)_{\sigma_1} \delta L^2_{\sigma_1} + \tau_{\mu\nu}(i) \tag{5.4a}$$

with the constants $\tau_{\mu\nu}(i)$ chosen so that $0 = \delta L^2$ for all the other legs in $S(\sigma_1)$ and

$$0 = (\Delta x^\mu \Delta x^\nu)_{\sigma_1} \tau_{\mu\nu}(i)\ . \tag{5.4b}$$

The $\delta g_{\mu\nu}$ are identically zero outside of $S(\sigma_1)$. Any other choice will lead to unacceptable changes in the geometry (ie. dislocations or fractures in the simplicies). Unfortunately this set of $\delta g_{\mu\nu}$ are not infinitely differentiable and so they cannot be used directly as test functions. However since the support of $\delta g_{\mu\nu}$ is compact it is always possible [6,7] to construct the test functions $\delta\overline{g}_{\mu\nu}$ that have the same values as $\delta g_{\mu\nu}$ in $S(\sigma_1)$ except at the points where $\delta g_{\mu\nu}$ is discontinuous. However, from (5.4a,b) one has

$$\delta L^2_{\sigma_1} = (\Delta x^\mu \Delta x^\nu)_{\sigma_1}\ \delta g_{\mu\nu}(i)\ .$$

The left hand side is constant throughout $S(\sigma_1)$ and therefore

$$\delta L^2_{\sigma_1} = (\Delta x^\mu \Delta x^\nu)_{\sigma_1}\ \delta\overline{g}_{\mu\nu}$$

no matter what choice is made for $\delta\overline{g}_{\mu\nu}$ at the points of discontinuity of $\delta g_{\mu\nu}$. Finally notice that since the $\delta\overline{g}_{\mu\nu}$ are constant in any neighbourhood of the σ_2's in $S(\sigma_1)$ then they must also be constant on those σ_2's.

Using the simple relation $\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$ the previous expression may also be written as

$$\delta L^2_{\sigma_1} = -\left(\Delta x_\mu \Delta x_\nu\right)_{\sigma_1} \delta\overline{g}^{\mu\nu} \ . \tag{5.5}$$

Now set $f^{\mu\nu} = \delta\overline{g}^{\mu\nu}$ and combine the above equations (4.6,5.5) to obtain

$$\overline{f}(\sigma_2, \sigma_1) = -\delta L^2_{\sigma_1} \ .$$

This shows that $\overline{f}$ is independent of σ_2. This then allows one to re-arrange the terms in (4.7) to obtain

$$I_3(\delta\overline{g}^{\mu\nu}) = 2 \sum_{\sigma_2(\sigma_1)} \Delta\theta \, \frac{\partial A}{\partial L} \, \delta L_{\sigma_1}$$

where the sum includes only those bones that are attached to this leg. This shows clearly that for this choice of variations in the metric

$$\delta I = I_3(\delta\overline{g}^{\mu\nu}) = \delta I' \ . \tag{5.6}$$

This result is also not un-expected. Having shown that $I = I'$ for any weak spacetime it follows that $\delta I = \delta I'$ for small variations in the metric.

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Appendix A.

The boundary terms in (4.3) may be simplified by first writing

$$
\begin{aligned}
H_\perp &= H^{\mu\nu} n_\mu n_\nu \\
&= \left(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\nu}\eta^{\alpha\beta}\right) n_\mu n_\nu \gamma_{\alpha\beta} \\
&= \left(n^\mu n^\nu - n^\alpha n_\alpha \eta^{\mu\nu}\right) \gamma_{\mu\nu} \\
&= \left(n^\mu n^\nu - n^\alpha n_\alpha g^{\mu\nu}\right) \gamma_{\mu\nu} + O(\gamma^2)\,.
\end{aligned}
$$

But $g^{\mu\nu} n^\alpha n_\alpha - n^\mu n^\nu$ are the contravariant components of the induced metric on the face σ_3. Denote these components by $h^{\mu\nu}$. Let $\overline{h}_{\mu\nu}$ and $\overline{n}_\mu$ be the respective values of $h_{\mu\nu}$ and n_μ in the unperturbed spacetime. Then

$$\eta_{\mu\nu} = \overline{h}_{\mu\nu} + \overline{n}_\mu \overline{n}_\nu$$

and

$$g_{\mu\nu} = h_{\mu\nu} + n_\mu n_\nu = \eta_{\mu\nu} + \gamma_{\mu\nu}$$

Consequently

$$
\begin{aligned}
H_\perp &= -h^{\mu\nu}\gamma_{\mu\nu} \\
&= -h^{\mu\nu}\Delta h_{\mu\nu}
\end{aligned}
$$

where $\Delta h_{\mu\nu} = h_{\mu\nu} - \overline{h}_{\mu\nu}$ and terms second order in γ have been discarded. Now let

$$
\begin{aligned}
h &= \det(h_{\mu\nu}) \\
\Delta h &= \det(h_{\mu\nu}) - \det(\overline{h}_{\mu\nu})
\end{aligned}
$$

then

$$H_\perp = -\frac{\Delta h}{h}\,.$$

This last quantity can also be obtained by computing the fractional change in the 3-volume of this face. Let $V(\sigma_3)$ be the 3-volume of this face, then

$$V(\sigma_3) = k\sqrt{h}$$

for some simple constant k (ie. k is independent of the leg-lengths). So finally one obtains

$$H_\perp = -\frac{1}{2}\frac{\Delta V(\sigma_3)}{V(\sigma_3)} \ .$$

The contribution from the boundary terms in (4.3) may now be written as

$$B.T.'s = -\sum_{\sigma_3(\partial M)} \frac{1}{2}\frac{\Delta V(\sigma_3)}{V(\sigma_3)} \int_{\sigma_3} \frac{df}{dn}\, d^3V \ .$$

Appendix B.

The simple identity

$$b^{\mu\nu}(\sigma_2) = \frac{1}{A(\sigma_2)} \sum_{\sigma_1(\sigma_2)} \frac{1}{L}\frac{\partial A}{\partial L}\, \Delta x^\mu \Delta x^\nu \qquad (B.1)$$

is rather easy to prove. First write

$$A(\sigma_2) = k'\sqrt{b}$$

where k' is some simple constant and $b = \det(b_{\mu\nu})$. Then

$$\frac{\delta A}{A} = \frac{1}{2}\frac{\delta b}{b} \ . \qquad (B.2)$$

However one also has

$$\delta b = b\, b^{\mu\nu}\delta b_{\mu\nu} \ ,$$

$$\delta A = \sum_{\sigma_1(\sigma_2)} \frac{\partial A}{\partial L}\,\delta L \ ,$$

and

$$\delta L^2 = \Delta x^\mu \Delta x^\nu \, \delta b_{\mu\nu} \ .$$

A substitution of these formulae into (B.2) and a subsequent comparison of the coefficients of $\delta b_{\mu\nu}$ on each side of the equation will lead to the stated identity (B.1).

Appendix C.

It was claimed in section § 5 that the only acceptable $\delta g^{\mu\nu}$ were those that were constant in each of the 4-simplicies attached to a specific leg. Rather than proving this statement directly it is easier to prove a related statement for a pair of adjacent triangles in 2-dimensions.

For the moment consider just one triangle with the vertices $\sigma_0(i), i = 1, 2, 3$. It is convenient to choose the coordinates such that

$$\begin{aligned} x^\mu(1) &= (1,0)^\mu \ , \\ x^\mu(2) &= (0,1)^\mu \ , \\ x^\mu(3) &= (0,0)^\mu \ . \end{aligned}$$

The associated $g_{\mu\nu}$ can then be calculated from (1.1). Our aim is to find a set of small perturbations $\delta g_{\mu\nu}(x^\alpha)$ that transforms this triangle into some other, slightly perturbed, triangle. This will be achieved by demanding that the triangles with coordinates $x^\mu(i), i = 1, 2, 3$ and $\lambda x^\mu(i), i = 1, 2, 3$ are congruent for any small perturbation and for any $0 \le \lambda \le 1$.

The vertices of the introduced (ie. the interior) triangle will be denoted by $\sigma_0(i'), i = 1, 2, 3$ with $\sigma_0(3') = \sigma_0(3)$ and with $\sigma_0(1')$ lying somewhere between $\sigma_0(1)$ and $\sigma_0(3)$. Let y^μ and y'^μ be the coordinates for points on the legs with vertices $\sigma_0(1), \sigma_0(2)$ and $\sigma_0(1'), \sigma_0(2')$ respectively. A convenient parameterization is

$$\begin{aligned} y^\mu &= t\, x^\mu(2) + (1-t)\, x^\mu(1) \ , \\ y'^\mu &= \lambda y^\mu \end{aligned}$$

where $0 \le t \le 1$. Then

$$L_{12} + \Delta L_{12} = \int_0^1 \left\{ \Big(g_{\mu\nu} + \delta g_{\mu\nu}(t, 1-t) \Big) \dot{y}^\mu \dot{y}^\nu \right\}^{1/2} \, dt \; ,$$

$$L_{1'2'} + \Delta L_{1'2'} = \lambda \int_0^1 \left\{ \Big(g_{\mu\nu} + \delta g_{\mu\nu}(\lambda t, \lambda - \lambda t) \Big) \dot{y}^\mu \dot{y}^\nu \right\}^{1/2} \, dt \; .$$

However, if the triangles are to be congruent then

$$L_{1'2'} = \lambda L_{12} \; ,$$

$$L_{1'2'} + \Delta L_{1'2'} = \lambda \left(L_{12} + \Delta L_{12} \right) \; .$$

These conditions are certainly satisfied when

$$\delta g_{\mu\nu}(t, 1-t) = \delta g_{\mu\nu}(\lambda t, \lambda - \lambda t)$$

whenever $0 \le t \le 1$ and $0 \le \lambda \le 1$. This requires $\delta g_{\mu\nu}$ to be constant along each radial ray from $\sigma_0(3)$. Since the choice of $\sigma_0(3)$ is arbitrary it follows that $\delta g_{\mu\nu}$ must be constant throughout the triangle.

Now suppose that there is a second triangle and that it is attached to the vertices $\sigma_0(1)$ and $\sigma_0(2)$. The problem now is to find the set of $\delta g_{\mu\nu}$'s that correspond to variations in only one leg-length L_{12}. The $\delta g_{\mu\nu}$ must also be constant inside the second triangle but the constants may differ from those in the first triangle. Let $\delta g_{\mu\nu}(i), i = 1, 2$ be the values of $\delta g_{\mu\nu}$ in the pair of triangles. Then it is always possible to write

$$\delta g_{\mu\nu}(i) = \left(\Delta x_\mu \Delta x_\nu \right)_{12} \delta L_{12}^2 + \tau_{\mu\nu}(i) \qquad (C.1)$$

for some set of small constants $\tau_{\mu\nu}(i), i = 1, 2$ that satisfy

$$0 = \tau_{\mu\nu} \left(\Delta x^\mu \Delta x^\nu \right)_{12} \; . \qquad (C.2)$$

(The remaining conditions needed to fully determine the $\tau_{\mu\nu}$ are that the variations in all other leg-lengths must vanish). Its rather easy to see that the same equations (C.1,C.2) will arise when considering the full 4-dimensional case in which many 4-simplicies share the same leg. It is important to note that the first term in (C.1) is constant throughout all of the 4-simplicies attached to this leg.

Figures.

Fig. 1. A two dimensional cross-section of the pair of 4-simplicies $\sigma_4(1)$ and $\sigma''_4(1)$. The cross-section is taken in the plane perpendicular to the bone σ_2. The transformation $\Lambda^\mu{}_\nu$ must map the points a and b into the points a'' and b'' respectively. This involves a rotation and a (radial) dilation.

Fig. 1.

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