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PARTICLE PATHS IN A SCHWARZSCHILD SPACETIME VIA THE REGGE CALCULUS

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As a test of the Regge calculus Williams and Ellis [1,2] computed, among other quantities, the precession of the perihelia of Mercury. Unfortunately they did not obtain anywhere near the correct values. Their results varied between -2.74 and 0.42 radians per orbit. Their best result was 9.1×10^{-4} radians per orbit whereas the correct analytic value is 5.0×10^{-7} radians per orbit. A continuous time version of their equations will be presented. It will be shown numerically that, for a sufficiently fine discretization, the global error between the Regge and Schwarzschild geodesics varies linearly with the typical length scale for the Regge simplices. Some simple modifications to the continuous time equations will then be presented. It will be shown both numerically and analytically that the modified equations yield paths that converge quadratically to the Schwarzschild geodesics. The modified Regge equations will then be applied to the problem of computing the precession of the perihelia of Mercury. The result is a precession of 5.0×10^{-7} radians per orbit.

1. Introduction.

It has often been argued that the Regge Calculus should be an excellent tool for the construction and subsequent investigation of numerical spacetimes. Although some considerable effort has gone into demonstrating the first point, very little has been done in establishing (or refuting) the second point. A notable exception is the work of Williams and Ellis [1,2]. They choose to investigate a given Regge spacetime by examining the trajectories of particles and photons. They developed a general formalism applicable to Regge spacetimes built from fully four dimensional blocks. Their basic idea was to use the fact that any pair of adjacent blocks can be covered by a flat Minkowskian metric. This means that the geodesic in any one block is a straight line segment and that the propagation of that geodesic into an adjacent block is obtained by a direct extension of that line segment.

Their main equations can be described as follows. Suppose a pair of adjacent blocks are denoted by σ and $\overline{\sigma}$. Coordinates are chosen in each block and will be denoted by x^μ and $\overline{x}^\mu$ respectively. As is customary in the Regge calculus these coordinate frames are chosen so that the metric components are constant inside each block. In such cases the two neighbouring frames will be related by a transformation of the form

$$\overline{x}^\mu = A^\mu{}_\nu x^\nu + B^\mu \tag{1.1}$$

for some choice of constants $A^\mu{}_\nu$ and B^μ. This transformation can also be viewed as a series of translations, rotations and boosts. The constants $A^\mu{}_\mu$ and B^μ can be determined by the known leg-lengths in each block and the requirement that the metric be flat on the interior of the pair of blocks. Let the paths in each block be represented by $x^\mu(s)$ and $\overline{x}^\mu(s)$ where s is an affine parameter (usually the proper distance). In this system of coordinates the components of the tangent vectors in each block will be constant and will be denoted by v^μ and $\overline{v}^\mu$. The essence of the the Williams and Ellis procedure is as follows.

- Determine the $A^\mu{}_\nu$ and B^μ for the pair of blocks.
- Choose an initial point x^μ_0 and a constant in v^μ in σ.
- Extend the straight line $x^\mu(s) = sv^\mu + x^\mu_0$ up to the interface $\sigma \cap \overline{\sigma}$.

- Use the coordinate transformation (1.1) to compute $\overline{v}^\mu$ and $\overline{x}^\mu(s)$ at this point in $\overline{\sigma}$.
- Repeat the process for any subsequent blocks.

Williams and Ellis demonstrated this procedure on a number of simple problems in a Schwarzschild spacetime. The blocks of their Regge spacetime were chosen as coordinate cubes (ie. the edges of each Regge block were aligned along the t, r, θ, ϕ coordinate curves). The accuracy of such Regge spacetimes had previously been demonstrated by Wong [3] and therefore any errors incurred in tracking a true geodesic in the Schwarzschild spacetime with the Regge geodesic could only be attributed to the method of constructing the Regge geodesic. Of course the hope was that since the Regge metric was an accurate representation of the Schwarzschild metric then the Regge geodesics should also be an accurate representation of the Schwarzschild geodesics. This seems very plausible when one takes a global view of the geodesics. That is, given a pair of points P and Q the path which extremizes the distance between P and Q must be a geodesic. Since the metrics of the two spacetimes are similar one expects the geodesics to be similar. But if one takes a local formulation of geodesics, a path generated by successively parallel transporting its tangent vector, it is then not so obvious that the two geodesics should remain close (any small errors may accumulate along the path). It is this later approach which the above procedure follows. Consequently the issue of accuracy in this method is non-trivial. Fortunately, though, most of Williams and Ellis's calculations appeared to suggest that as the size of the blocks were reduced the Regge geodesics converged towards the Schwarzschild geodesics. They showed this for particles and photons falling radially and for some (approximately) circular particle orbits. One of the limitations of their computations was the small number of cases they studied. In the radial infall problem they calculated the geodesics for 12 cases with 90 and 180 radial blocks, 30 and 300 angular sub-divisions and four choices for the time step. There does not appear to be a clear and consistent pattern of convergence in their results. It is true that their results differ by only a fraction of a percent (in some cases) from the correct result but they did not establish how that error scales with the number or size of the blocks. It is necessary, for convergence, to show that the global error is of order no larger than the largest blocks size. Such a pattern can not be seen in their results.

Most of this paper is concerned with this issue, the nature of the convergence of the Regge calculus geodesics to the Schwarzschild geodesics. Another important issue, which was not discussed by Williams and Ellis, is how to choose the map by which paths in the Regge spacetime are mapped into the Schwarzschild spacetime. Thus if an error is found when comparing the Regge and Schwarzschild paths should it be attributed to the choice of the spacetime map or to the method by which the Regge paths were constructed?

The formulation presented by Williams and Ellis assumed that both space and time were fully discretized. However the current trend today in numerical relativity is to discretize space while retaining a continuous time (the continuous time “3+1” approach). This is the approach to be taken in this paper.

The details for the construction of the Regge spacetimes will be presented in the following section. The equations of motion for a discrete time will then be derived. The first departure from the Williams and Ellis approach will then be taken – the length of the time like edges of the Regge cubes will be reduced to zero. This will be done in section 3 and will lead to a set of coupled ordinary differential equations. Some numerical solutions of these equations will be presented in section 4. It will be shown that these solutions appear to converge linearly to the correct Schwarzschild solutions. For generic spacetimes and for certain problems this linear convergence may not be satisfactory. Thus in section 4 a modified set of Regge equations will be presented. It will be shown that these equations have quadratic convergence properties. These equations will be applied in section 5 to the problem of predicting the precession of the perihelia of Mercury. It will be shown that our results are in very close agreement with the analytical value.

2. Construction of the Regge spacetime.

The Schwarzschild spacetime will be the example upon which our ideas will be presented. Since it is already known (Wong [3]) that the solution of the Regge equations yields an accurate approximation to this spacetime it will not be necessary to solve the Regge equations. Instead the approach chosen by Williams and Ellis, to directly discretize the Schwarzschild spacetime, will be followed here.

Since the particle paths are known to lie in a fixed plane, it is sensible to assume at the outset that the motion takes place in the $\phi = \pi/2$ plane (where ϕ is the usual azimuthal angle in the Schwarzschild coordinates).

The construction of the Regge spacetime is quite simple. First sub-divide the $\phi = \pi/2$ plane by drawing a set of coordinate curves as indicated in figure (2.1). Each tile in this figure will be replaced with a tile in the Regge spacetime, the dimensions of the Regge tile being computed from the Schwarzschild tile. The discretization in time is obtained by stacking many copies of figure (2.1) in a “2+1” spacetime split as indicated in figure (2.2). Again the lengths of the time like legs are calculated from the Schwarzschild spacetime. In the Regge spacetime the edges of each “2+1” cube are dictated to be straight line segments and the components of the metric are chosen as constants. It is not hard to see that the faces of the cubes are planar.

Let the successive values of r on each of the circles by denoted by $r_i, i = 0, 1, 2 \cdots N_r$ with $r_0 > 2m$. As the curvature can be significantly larger near $r = 2m$ than for $r \gg 2m$ it is wise to assume that $r_{i+1} - r_i$ is not a constant. In contrast it is safe to choose constant step lengths for θ and t. Thus set $\theta_j = (2j-1)\Delta\theta, j = 1, 2, 3 \cdots N_\theta$ with $\Delta\theta = \pi/N_\theta$ and $t_j = j\Delta t, j = 0, 1, 2 \cdots N_t$. In fact, since the Schwarzschild spacetime is static it is sufficient (in this gauge) to set $N_t = 2$ and to use a periodic boundary condition on t.

The Regge tile corresponding to the Schwarzschild tile with lower left corner at $(r = r_i, \theta = \theta_j)$ will be represented by σ_{ij}. The Regge cube built on this tile will be denoted by Σ_{ij}. The tile and cube lying in the next time-slice above σ_{ij} and Σ_{ij} will be denoted by $\overline{\sigma}_{ij}$ and $\overline{\Sigma}_{ij}$ respectively.

It is very convenient in the following calculations to have a local coordinate frame in each Regge cube. Let the Regge coordinates be (τ, u, v) with the v-axis chosen along the inner r=constant edge, the u-axis through the centre of the spatial tile and the τ-axis chosen to be orthogonal to the (u, v)-axes. This arrangement is shown in figure (2.3). Note that Williams and Ellis chose a different frame – they placed the origin of their frame at the centre of the cube.

The metric in each Regge cube is chosen to be

$$ds^2 = -d\tau^2 + du^2 + dv^2 \tag{2.1}$$

whereas the metric for each Schwarzschild cube is

$$ds^2 = -\left(1 - \frac{2m}{r}\right)(dt)^2 + \frac{(dr)^2}{1 - \frac{2m}{r}} + (r\,d\theta)^2 \tag{2.2}$$

Let n_i be defined by

$$n_i = (1 - \frac{2m}{r_i})^{1/2} \tag{2.3}$$

The dimensions (measured along the local coordinate axes) of the Regge cube can now be calculated from

$$\tau_i = n_i \Delta t \tag{2.4a}$$

$$u_i = 2\left(\frac{r_{i+1} - r_i}{n_{i+1} + n_i}\right) \tag{2.4b}$$

$$v_i = \frac{2\pi r_i}{N_\theta} \tag{2.4c}$$

Previous experience has shown that it is best to construct the tiles so that the edges are roughly of equal length. That is, there should be no long and skinny or fat and squat tiles. This is easily achieved by choosing $N_\theta = O(N_r)$. The errors incurred by this simplicial approximation can then (in principle) be expressed in terms of N_r. An equally useful parameter for such purposes is the size of the largest diagonal over all of the tiles. This will be denoted by Δ. Thus the limit $N_r \to \infty$ is equivalent to requiring $\Delta \to 0$.

The coordinates of a point on the particle path in the Regge cube Σ_{ij} will be denoted by $x^\mu_{ij}(s)$ where s is the proper time measured along the path. The associated velocity 4-vector will be represented by $v^\mu_{ij}(s) = dx^\mu_{ij}(s)/ds$. In the Schwarzschild spacetime the position and velocity 4-vector will be denoted by $y^\mu(s)$ and $w^\mu(s) = dy^\mu(s)/ds$ respectively. Similar definitions apply in the future cube $\overline{\Sigma}_{ij}$.

3. The continuous time equations of motion.

Our aim is to develop appropriate formulae for the particle paths in the Regge spacetime **without** reference to the parent Schwarzschild spacetime. This is essential if this method is to be used for generic Regge spacetimes. Thus, for the moment, attention will be confined to just the Regge spacetime. Later in this paper the Schwarzschild spacetime will be re-introduced for the purposes of comparison.

In each Regge cube the particle path will be of the form $x^\mu(s) = sv^\mu + x^\mu_0$. The only real issue to be resolved now is how to propagate this path into the adjacent cubes. As outlined above, this amounts to performing a coordinate transformation between adjacent cubes. For our choice of cube there are three cases to consider. The cases are characterized by the type of the face the path terminates on, the τ=constant, u=constant or θ=constant faces (this throw-back to the Schwarzschild spacetime is only for ease of description). The three cases are illustrated in figures (3.1a,b,c). The simple nature of the Schwarzschild spacetime, static and zero extrinsic curvature in this gauge, greatly simplifies the calculations. The coordinate transformations can almost be written down by inspection.

- u=constant

Since the extrinsic curvature of each τ=constant slice is zero it follows that the two adjacent frames, for cubes in the same τ=constant slice, must be related by spatial translations and spatial rotations. In this case it is easy to see that the only transformation required is a shift along the u-axis. Thus

$$\begin{aligned} \tau_{(i-1)j} &= \tau_{ij} \\ u_{(i-1)j} &= u_{ij} + u_{i-1} \\ v_{(i-1)j} &= v_{ij} \end{aligned} \tag{3.1}$$

- θ=constant

In this case the coordinate transformation entails a shift along the $v-$axis, a rotation in the (u,v)-plane followed by another shift along the v-axis. The result is

$$\begin{aligned} \tau_{i(j+1)} &= \tau_{ij} \\ u_{i(j+1)} &= \cos(2\Delta\overline{\theta})u_{ij} + \sin(2\Delta\overline{\theta})(v_{ij} - \frac{1}{2}v_i) \\ v_{i(j+1)} &= \cos(2\Delta\overline{\theta})(v_{ij} - \frac{1}{2}v_i) - \sin(2\Delta\overline{\theta})u_{ij} - \frac{1}{2}v_i \end{aligned} \tag{3.2}$$

where

$$\Delta\overline{\theta} = \mathrm{Tan}^{-1}\left(\frac{v_{i+1} - v_i}{2u_i}\right)$$

- τ=constant

This is similar to the previous case – a shift along the τ-axis, a boost in the (τ,u)-plane and a further shift, this time along the $\overline{\tau}$-axis. The result is

$$\begin{aligned} \overline{\tau}_{ij} &= \cosh(2\Delta\beta_i)(\tau_{ij} - \frac{\tau_i}{2}) - \sinh(2\Delta\beta_i)u_{ij} - \frac{\tau_i}{2} \\ \overline{u}_{ij} &= \cosh(2\Delta\beta_i)u_{ij} - \sinh(2\Delta\beta_i)(\tau_{ij} - \frac{\tau_i}{2}) \\ \overline{v}_{ij} &= v_{ij} \end{aligned} \tag{3.3}$$

where the boost parameter $\Delta\beta_i$ is given by

$$\Delta\beta_i = \mathrm{Tanh}^{-1}\left(\frac{\tau_{i+1} - \tau_i}{2u_i}\right)$$

Our aim is to produce a continuous time formulation by letting $\Delta t \to 0$. Thus, neglecting terms of order $(\Delta t)^3$ and higher, this equation can also be written as

$$\Delta\beta_i = \frac{\dot\beta_i\,\tau_i}{2}$$

where $\dot\beta_i$ is defined by

$$\dot\beta_i = \left(\frac{\frac{\tau_{i+1}}{\tau_i} - 1}{u_i}\right) = \left(\frac{\frac{n_{i+1}}{n_i} - 1}{u_i}\right) \tag{3.4}$$

Notice that in any continuous time formulation of the Regge calculus the lapse function will be known (either on the vertices or at the centres of the blocks). Thus the $\dot\beta_i$ can be computed in any generic Regge spacetime.

Recall that the particle path in Σ_{ij} has coordinates $x^\mu_{ij}(s)$ and velocity 4-vector $v^\mu_{ij}(s) = dx^\mu_{ij}(s)/ds$ where s is the proper time measured along the particle path. When Δt is much smaller than both u_i and v_i this path will most probably pass through the τ=constant roof of the cube. In which case it follows from equations (3.3) that

$$\begin{aligned}
\overline{v}^\tau_{ij} - v^\tau_{ij} &= -\dot\beta_i\, v^u_{ij}\,\tau_i\\
\overline{v}^u_{ij} - v^u_{ij} &= -\dot\beta_i\, v^\tau_{ij}\,\tau_i\\
\overline{v}^v_{ij} - v^v_{ij} &= 0
\end{aligned}$$

In preparing to reduce these equations to a set of differential equations it is important to view the $\overline{v}^\mu_{ij}$ and v^μ_{ij} as the values of a smooth field $\tilde v^\mu_{ij}(s)$ at $\overline\tau = 0$ and $\tau = 0$ respectively. This situation is shown in figure (3.2). Suppose the proper time s is measured from the point Q, then the correctly centred approximations are

$$\overline{v}^\mu_{ij} = \tilde v^\mu_{ij} + \frac{d\tilde v^\mu_{ij}}{ds}\frac{\Delta s}{2} + O(\Delta s)^2$$

$$v^\mu_{ij} = \tilde v^\mu_{ij} - \frac{d\tilde v^\mu_{ij}}{ds}\frac{\Delta s}{2} + O(\Delta s)^2$$

where Δs is the proper time measured along the particle path from P to R. This is given, to leading order in Δt, by

$$\Delta s = \Gamma \left(\frac{1}{v^\tau_{ij}} + \frac{1}{\bar{v}^\tau_{ij}} \right) \frac{\tau_i}{2}$$

where

$$\Gamma = \frac{(n_{i+1} + n_i) u_i}{2\,(n_i u_i + (n_{i+1} - n_i) u)}$$

In all of the subsequent numerical experiments the factor Γ is found to equal 1 to within $10^{-5}\%$. In fact, it is easy to show that

$$\frac{n_i}{n_{i+1}} - 1 < 2(\Gamma - 1) < \frac{n_{i+1}}{n_i} - 1$$

and thus $\lim_{N_r \to \infty} \Gamma = 1$ for $r > 2m$. Thus at this point Γ will be set equal to 1. The above equations can now be combined and the limit $\Delta t \to 0$ taken, with the result (after replacing $\tilde{v}$ with v)

$$\frac{dv^\tau_{ij}}{ds} = -\dot{\beta}_i \, v^\tau_{ij} \, v^u_{ij} \tag{3.5a}$$

$$\frac{dv^u_{ij}}{ds} = -\dot{\beta}_i \, (v^\tau_{ij})^2 \tag{3.5b}$$

$$\frac{dv^v_{ij}}{ds} = 0 \tag{3.5c}$$

These are the basic particle path equations for this simple continuous time Regge spacetime. They apply in each tile σ_{ij}. The equations (3.1,3.2) can used to propagate the solution across the interfaces between adjacent tiles. The unlikely situation where the path bumps into a corner of the tile can be dealt with by making an arbitrary small perturbation of the path. The same strategy was adopted by Williams and Ellis.

For the sake of completeness here are the usual Schwarzschild geodesic equations.

$$n \frac{dw^t}{ds} = -2 n' w^t w^r \tag{3.6a}$$

$$\frac{1}{n}\frac{dw^r}{ds} = \frac{n'}{n^2}(w^r)^2 - n'n^2(w^t)^2 + rn(w^\theta)^2 \tag{3.6b}$$

$$r\frac{dw^\theta}{ds} = -2w^r w^\theta \tag{3.6c}$$

where $n^2 = 1 - 2m/r$ and $n' = dn/dr$.

4. The numerical experiments.

The numerical integration of (3.5) is trivial to implement. The most important question in this paper can now be put : How accurate is the Regge method? There is a lot more to this question than just accuracy. How are the Regge and Schwarzschild paths to be compared? Indeed how can one ensure that the same initial conditions are used for both the Regge and Schwarzschild paths? The Regge and Schwarzschild spacetimes are **not** the same since they clearly have metrics which can not be related by a smooth global coordinate transformation. Arguably the best that can be done is to use a map between the two spacetimes and hope that the associated errors do not dominate over whatever errors might exist between the Regge and Schwarzschild paths. The language here is necessarily imprecise because of the difficulty of clearly separating the errors introduced by the spacetime map from those inherent in the approximation of the Schwarzschild path with the Regge path. The hope of course is that the errors will diminish as the number of radial and angular sub-divisions is increased.

4.1 The spacetime map.

So we are lead to ask what quantities need to be mapped and what properties should the map possess? Since the initial data for each geodesic consists of $x^\mu, \dot{x}^\mu$ in the Regge spacetime and $y^\mu, \dot{y}^\mu$ in the Schwarzschild spacetime it follows that the map must act on this data. Futhermore, since the path within a tile should be continuous it follows that the map should also be continuous. Finally, the map should be one to one and invertible so that one can freely swap between the Regge to Schwarzschild spacetimes without loosing information.

Let $\mathcal{R}$ be the phase space for particle paths in the Regge spacetime. The coordinates in $\mathcal{R}$ can be chosen as $(x^\mu, \dot{x}^\mu)$. Similarly let $\mathcal{S}$ be the phase space, with coordinates $(y^\mu, \dot{y}^\mu)$, for particle paths in the Schwarzschild spacetime. The map $\Phi : \mathcal{R} \mapsto \mathcal{S}$ will be denoted by

$(y^\mu, \dot{y}^\mu) = (\phi(x^\mu), \phi^\star(x^\mu, \dot{x}^\mu))$ where ϕ and $\phi^\star$ are a pair of functions. Note that $\phi^\star$ need not be $d\phi/ds$. The above properties of the map can now be stated as requiring ϕ and $\phi^\star$ to be one to one, invertible and continuous functions within in each tile.

Consider a typical Regge tile σ_{ij}. It can be viewed as a truncated isosceles triangle as shown in figure (4.1). Consider now three coordinate systems in the tile. The first is the original local coordinates (u, v). The second is a polar coordinate system defined by $(\overline{r}, \overline{\theta})$. The third coordinate system $(\tilde{r}, \tilde{\theta})$ is formed by re-scaling the $(\overline{r}, \overline{\theta})$ so as to produce a Schwarzschild like angular coordinate $\tilde{\theta}$. As each of the coordinates $(u, v), (\overline{r}, \overline{\theta})$ and $(\tilde{r}, \tilde{\theta})$ are defined relative to the tile σ_{ij} each coordinate should be written with an ij subscript. To do so throughout the following would lead to some very awkward notation. Thus this subscript will not be explicitly included.

In each of these coordinate systems the metric is known to be flat and thus may be written as

$$(ds)^2 = (du)^2 + (dv)^2 = (d\overline{r})^2 + \left(\overline{r}d\overline{\theta}\right)^2 = \left(\frac{d\tilde{r}}{\tilde{n}}\right)^2 + \left(\tilde{r}d\tilde{\theta}\right)^2$$

where $\tilde{n}$ is some (yet to be chosen) constant in σ_{ij}. Since the Regge metric was constructed from the Schwarzschild metric it follows that $\tilde{n}$ should be a good approximation to the function $n(r) = (1 - 2m/r)^{1/2}$. This fact will be used later when enforcing the above continuity and differentiability conditions. Using the known Schwarzschild metric at this stage is not cheating since this extra knowledge in no way alters how the Regge geodesic is constructed. The Schwarzschild metric is being re-introduced only to help in comparing the Regge and Schwarzschild geodesics.

Integrating the metric along the various coordinate curves leads to the following relations

$$\overline{r}_i \Delta\overline{\theta} = \tilde{r}_i \Delta\tilde{\theta} \qquad (\overline{r}_i + u_i)\Delta\overline{\theta} = \tilde{r}_{i+1}\Delta\tilde{\theta}$$

$$u_i = \frac{\tilde{r}_{i+1} - \tilde{r}_i}{\tilde{n}} \qquad v_i = 2\overline{r}_i \tan\Delta\overline{\theta} \qquad v_{i+1} = 2(\overline{r}_i + u_i)\tan\Delta\overline{\theta}$$

From these relations it is easy to show that

$$\tilde{n} = \frac{\Delta\overline{\theta}}{\Delta\tilde{\theta}} \tag{4.1a}$$

$$\overline{r}_i = \frac{u_i v_i}{v_{i+1} - v_i} \tag{4.1b}$$

$$\Delta\overline{\theta} = \mathrm{Tan}^{-1}\left(\frac{v_{i+1} - v_i}{2u_i}\right) \tag{4.1c}$$

The constant $\tilde{n}$ is chosen in this tile so that $\Delta\tilde{\theta} = \pi/N_\theta$. This leads to

$$\tilde{n} = \frac{N_\theta}{\pi}\mathrm{Tan}^{-1}\left(\frac{v_{i+1} - v_i}{2u_i}\right) \tag{4.2}$$

The first coordinate transformation, $(u, v) \leftrightarrow (\overline{r}, \overline{\theta})$, is defined by

$$\overline{r}\sin\overline{\theta} = v \tag{4.3a}$$

$$\overline{r}\cos\overline{\theta} = \overline{r}_i + u \tag{4.3b}$$

and the second, $(\overline{r}, \overline{\theta}) \leftrightarrow (\tilde{r}, \tilde{\theta})$, by

$$\frac{\tilde{r}}{\tilde{n}} = \overline{r} \tag{4.4a}$$

$$\tilde{n}\tilde{\theta} = \overline{\theta} \tag{4.4b}$$

Combining the two transformations yields

$$u = -\overline{r}_i + \frac{\tilde{r}}{\tilde{n}}\cos\left(\tilde{n}\tilde{\theta}\right) \tag{4.5a}$$

$$v = \frac{\tilde{r}}{\tilde{n}}\sin\left(\tilde{n}\tilde{\theta}\right) \tag{4.5b}$$

Differentiating the above leads to

$$\frac{du}{ds} = \frac{1}{\tilde{n}}\frac{d\tilde{r}}{ds}\cos\left(\tilde{n}\tilde{\theta}\right) - \tilde{r}\sin\left(\tilde{n}\tilde{\theta}\right)\frac{d\tilde{\theta}}{ds} \tag{4.5c}$$

$$\frac{dv}{ds} = \frac{1}{\tilde{n}}\frac{d\tilde{r}}{ds}\sin\left(\tilde{n}\tilde{\theta}\right) + \tilde{r}\cos\left(\tilde{n}\tilde{\theta}\right)\frac{d\tilde{\theta}}{ds} \tag{4.5d}$$

An equation for $d\tilde{t}/ds$ can be obtained by normalizing the velocity 4-vector. The result is

$$\tilde{n}\frac{d\tilde{t}}{ds} = \frac{d\tau}{ds} \tag{4.5e}$$

This equation also serves the dual purpose of defining, by an integration along the particle path, the transformation $\tilde{t} = \tilde{t}(\tau, u, v)$. This transformation will of course be path dependent.

The map between the Regge and Schwarzschild spaces can now be defined as follows. Consider a tile in the Regge spacetime and its corresponding tile in the Schwarzschild spacetime. Choose $(\tilde{r}, \tilde{\theta}, d\tilde{r}/ds, d\tilde{\theta}/ds)$ as coordinates for the Regge phase space $\mathcal{R}$ and $(r, \theta, dr/ds, d\theta/ds)$ as coordinates for the Schwarzschild phase space $\mathcal{S}$. The map Φ in these coordinates is then just the identity map, ie. $r = \tilde{r}, \theta = \tilde{\theta}$ etc.

The equations (4.1-4.5) fully describe the map between the Regge and Schwarzschild spacetimes.

Note that since $\tilde{n}$ is not continuous across the interfaces between neighbouring tiles, this map can introduce spurious bumps in the image of the Regge path in the Schwarzschild spacetime. Rather than attempting to overcome this problem at this stage it is instructive to review the results that the above map gives.

4.2. Results. I

Three particular experiments were performed. The first was for a particle released from rest at $r = r_{max}$ and allowed to fall radially, along $\theta = 0$, until it reached $r = r_{min}$. The second experiment differed only in that the particle was released at the junction of two radial segments, ie. along $\theta = \Delta\theta/2$. In the final experiment the particle was again required to fall from $r = r_{max}$ to $r = r_{min}$ but starting with a non-zero angular velocity.

It is easy to see from (3.6) that a particle when released from rest will travel along a path with v =constant. For the case where $v = 0$, as in the first experiment, the particle, quite correctly, is seen to fall inwards along the radial line $\theta = 0$. However, for the cases where $v \neq 0$ the particle will eventually move into a different radial segment. Thus in such cases a particle falling from rest will develop a non-zero angular velocity! It was for this reason that the second experiment was proposed. It was found that the particle path oscillated (with increasing amplitude) about the line $\theta = \Delta\theta/2$.

In each experiment the initial data was first posed in the Schwarzschild spacetime and then mapped to the initial data in the Regge spacetime via equations (4.1-4.5). A fourth order Runge-Kutta routine, with a fixed time step, was then used to integrate the coupled systems (3.5) and (3.6). At each stage in the integration the Regge path was mapped back into the Schwarzschild spacetime. The errors, $e(y^\mu)$ and $e(\dot{y}^\mu)$, were then estimated as

$$\begin{aligned} e(y^\mu) &= y^\mu(s) - \phi\left(x^\mu(s)\right) \\ e(\dot{y}^\mu) &= \dot{y}^\mu(s) - \phi^\star\left(x^\mu(s), \dot{x}^\mu(s)\right) \end{aligned}$$

Ideally these errors should diminish to zero as N_r and N_θ are increased. An estimate of the error was also calculated according to

$$e = \frac{1}{N_{dt}} \sum_{i=1}^{i=N_{dt}} \left(\sum_{\mu=1}^{\mu=3} e(y^\mu)^2 \right)_i^{1/2}$$

where N_{dt} equals the number of time steps in the integration.

In each experiment the Regge spacetime was constructed with $r_0 = 10m, r_{N_r} = 110m, u_i = 100m/N_r, m = 1$. The r_i were calculated by a 2nd order accurate integration of $dr/du = \sqrt{(1 - 2m/r)}$. The values of N_r and N_θ were typically in the range of 100 to 12800. A constant time step was used in the integrations and was calculated as the minimum of two candidate time steps. The first time step was calculated so that the particle would traverses the smallest tile, that the particle was likely to encounter, in a given number of steps. The second time step was calculated in a similar way but for a fictitious particle travelling at $0.2c$. The number of steps was typically four to ten.

The results are listed in tables (1a-5a). Some interesting behaviour can be clearly seen. First note that the entries in the lower left and upper right corners of each table quickly settle to a non-zero error. This is not a surprise for such combinations represent, for example, a very fine sub-division in r but a very coarse sub-division in θ thus leaving a finite residual error. Thus, as already noted, best results occur when $N_\theta = O(N_r)$. These entries are concentrated around the main diagonal of each table. It is reasonable to expect that, for $N_\theta = kN_r$ and a fixed k, that the error should vary in a regular fashion as $N_r \to \infty$. However, even though the error does seem to vanish, it does so in a very erratic way. It seems reasonable to suspect that this behaviour arises from the discontinuities in the spacetime map. There may also be similar effects arising from the transformations of the data as the path crosses from one tile to another. Thus it seems prudent to investigate the nature of these discontinuities and their influence on the errors $e(y^\mu)$.

4.3. The modified equations of motion.

Consider three consecutive tiles $\sigma_{i(j-1)}, \sigma_{ij}, \sigma_{i(j+1)}$. For the moment let us consider just one of the three equations of motion, namely (3.5a). This equation, in the coordinates of the three tiles, is

$$\frac{dv^\tau_{i(j+1)}}{ds} = -\dot\beta_i\, v^\tau_{i(j+1)}\, v^u_{i(j+1)} \qquad \text{in} \quad \sigma_{i(j+1)}$$

$$\frac{dv^\tau_{ij}}{ds} = -\dot\beta_i\, v^\tau_{ij}\, v^u_{ij} \qquad \text{in} \quad \sigma_{ij}$$

$$\frac{dv^\tau_{i(j-1)}}{ds} = -\dot\beta_i\, v^\tau_{i(j-1)}\, v^u_{i(j-1)} \qquad \text{in} \quad \sigma_{i(j-1)}$$

These equations can be referred to the coordinates of σ_{ij} by way of the transformations (3.2) with the result

$$\frac{dv^\tau_{ij}}{ds} = -\dot\beta_i\, v^\tau_{ij}\left(v^u_{ij}\cos(2\Delta\bar\theta) + v^v_{ij}\sin(2\Delta\bar\theta)\right) \qquad \text{in} \quad \sigma_{i(j+1)}$$

$$\frac{dv^\tau_{ij}}{ds} = -\dot\beta_i\, v^\tau_{ij}\left(v^u_{ij}\right) \qquad \text{in} \quad \sigma_{ij}$$

$$\frac{dv^\tau_{ij}}{ds} = -\dot\beta_i\, v^\tau_{ij}\left(v^u_{ij}\cos(2\Delta\bar\theta) - v^v_{ij}\sin(2\Delta\bar\theta)\right) \qquad \text{in} \quad \sigma_{i(j-1)}$$

This shows clearly that the coefficients of the global Regge equation are not continuous at the interfaces between adjacent Regge tiles. Compare this with the Schwarzschild equations

(3.6)which are smooth and continuous everywhere (except at $r = 0$ and $r = 2m$). Thus it seems one improvement can be made by smoothing out the discontinuities in the Regge equations. The form of the above three equations suggest that an appropriate interpolation is

$$\frac{dv^{\tau}_{ij}}{ds} = -\dot{\beta}\, v^{\tau}_{ij} \left(v^{u}_{ij} \cos\tilde{\theta} + v^{v}_{ij} \sin\tilde{\theta} \right) \tag{4.6a}$$

where

$$\tan\tilde{\theta} = \frac{v(v_{i+1} - v_i)}{(u_i - u)v_i + uv_{i+1}}$$

The removal of the subscript i on $\dot{\beta}$ will be explained shortly. Following the same procedure for the remaining two equations (3.5b,c) leads to

$$\frac{dv^{u}_{ij}}{ds} = -\dot{\beta} \left(v^{\tau}_{ij} \right)^2 \cos\tilde{\theta} \tag{4.6b}$$

$$\frac{dv^{v}_{ij}}{ds} = -\dot{\beta} \left(v^{\tau}_{ij} \right)^2 \sin\tilde{\theta} \tag{4.6c}$$

When the same analysis is carried out for three consecutive radial tiles $\sigma_{(i-1)j}, \sigma_{ij}, \sigma_{(i+1)j}$ all that one obtains is that $\dot{\beta}$ should be continuous at the edges of the tiles. Thus the $\dot{\beta}$ in the above where taken as a linear interpolation of the $\dot{\beta}_i$ with the nodes chosen at the edges of each tile $u = 0, v = 0$ and $u = u_i, v = 0$. The particular interpolation used was

$$\dot{\beta}(u, v) = A + Bu \qquad\qquad 0 \le u \le \Delta u \tag{4.7}$$

where

$$A = \frac{\dot{\beta}_i + \dot{\beta}_{i-1}}{2}$$

$$B = \frac{\dot{\beta}_{i+1} - \dot{\beta}_{i-1}}{2\Delta u}$$

and $\Delta u = u_i$

Similar arguments for continuity can be applied to the spacetime map. It seems reasonable that if a path and its tangent vector are continuous in one spacetime then they should also be continuous in the other spacetime. By inspection of the equations (4.3-4.5) it is easy to see that this can be satisfied if the piecewise constant $\tilde{n}$ (given by (4.2)) is replaced by a linear interpolant. The interpolation is based on the nodes chosen at the centre of each tile $u = u_i/2, v = 0$ at which point the interpolant has the value $(n_{i+1} + n_i)/2$. It will be seen shortly that for other reasons it is in fact necessary to employ a higher order interpolant.

It is instructive at this stage to ask what differential equations does the image path $\phi\left(x^\mu(s)\right)$ satisfy in the Schwarzschild spacetime? Such equations should be comparable to the Schwarzschild geodesic equations and should help us in estimating the truncation error. To this end one can start with the modified Regge equations and map them to the Schwarzschild spacetime via the map Φ. The result is

$$\tilde{n}\frac{d\tilde{v}^t}{ds} = -\left(\tilde{n}' + \dot{\beta}\right)\tilde{v}^t\tilde{v}^r \tag{4.8a}$$

$$\frac{1}{\tilde{n}}\frac{d\tilde{v}^r}{ds} = \frac{\tilde{n}'}{\tilde{n}^2}\left(\tilde{v}^r\right)^2 - \dot{\beta}\left(\tilde{v}^t\right)^2 + \tilde{r}\tilde{v}^\theta\frac{d\overline{\theta}}{ds} \tag{4.8b}$$

$$\tilde{r}\frac{d\tilde{v}^\theta}{ds} = -\tilde{v}^r\tilde{v}^\theta - \frac{1}{\tilde{n}}\tilde{v}^r\frac{d\overline{\theta}}{ds} \tag{4.8c}$$

where $\tilde{n}' = d\tilde{n}/dr$.

The first point to note is that the coefficients in these equations are, for linear functions $\tilde{n}$, not continuous across the edges of the Schwarzschild tiles. Thus, to be consistent, one should choose an interpolation of $\tilde{n}$ which ensures continuity of both $\tilde{n}$ and $\tilde{n}'$ at the edges where $u = 0$ and $u = u_i$. This entails four conditions which can be easily satisfied by a cubic interpolant.

In our numerical experiments u_i did not depend upon i which allowed us to write

$$\tilde{n}(u,v) = A + Bu + Cu^2 + Du^3 \qquad\qquad 0 \le u \le \Delta u \tag{4.9}$$

where

$$A = n_i$$
$$B = \frac{n_{i+1} - n_{i-1}}{2\Delta u}$$

$$C = \frac{-n_{i+2} + 4n_{i+1} - 5n_i + 2n_{i-1}}{2(\Delta u)^2}$$

$$D = \frac{n_{i+2} - 3n_{i+1} + 3n_i - n_{i-1}}{2(\Delta u)^3}$$

and where $\Delta u = u_i$.

Notice that the use of a cubic interpolant for $\tilde{n}(u, v)$ is forced upon us not by considerations of the Regge spacetime itself but rather by issues solely concerned with the spacetime map. For the purpose of computing the Regge geodesic in the Regge spacetime it is sufficient to use an $\tilde{n}(u, v)$ which is defined either on the verticies or, for example, at the centre of each tile.

From (2.3,3.4,4.4b,4.9) it is easy to show that

$$\tilde{n}(u, v) = n(r) + O(\Delta^2)$$

$$\dot{\beta}(u, v) = n'(r) + O(\Delta^2)$$

$$\frac{d\overline{\theta}}{ds} = n(r)\frac{d\tilde{\theta}}{ds} + O(\Delta^2)$$

Thus it is clear that the exact Schwarzschild equations (3.6) and the above equations (4.8) differ only by terms of order $O(\Delta^2)$. Consequently their solutions, starting with the same initial data, should also differ by terms of order $O(\Delta^2)$. This establishes that the modified Regge paths are globally second order accurate approximations to the Schwarzschild geodesics.

The same analysis can be applied to the original formulation of the Regge equations (3.5) with the result that the error should be of order $O(\Delta)$.

The solution of

4.4. Results. II.

The results of the three experiments, using the modified Regge equations (4.6) together with the modified spacetime map (4.1b,c,4.5,4.7,4.9) are shown in tables (1b-5b). It is clear that the global errors do appear to vanish as $\Delta \to 0$. The analysis of the previous section suggests that the global error should behave as $e = a\Delta^\gamma$ where a is a constant and $\gamma = 2$. The convergence rate, γ, was estimated only for the third experiment, this being the most demanding of the three experiments. The results have been plotted in figure (4.2). Note that in figure (4.2) some extra data points have been added so as to provide a better estimate of γ. Each data point was generated with $N_r = N_\theta$. The initial and final data for each run was as used in table (5a,b). The jagged curve represents the data obtained from the original formulations of both the equations of motion and the spacetime map. The second curve, marked with the solid markers, was generated by combining the original equations with the modified spacetime map. This shows clearly that it is the discontinuities in the original spacetime map that is responsible for the erratic behaviour. It also shows that the original formulation of the Regge equations yields an order Δ accurate scheme. The final curve, marked with the open markers, represents the combination of the modified equations of motion and the modified spacetime map. It shows quite clearly that this modified scheme is 2^{nd} order accurate, as expected.

It was found that the second experiment no longer gave anomalous results. It is easy to establish from (4.6) that a particle in free fall will always travel along $\tilde{\theta}$ =constant in the Regge spacetime.

These numerical experiments support the above conclusion that the solutions of the modified Regge equations (4.6) coupled with the modified spacetime map (4.1b,c,4.5,4.7,4.9) provide globally second order accurate approximations to the Schwarzschild geodesics. It also shows that the original Regge geodesics converge linearly to the Schwarzschild geodesics.

5. The precession of the perihelia of Mercury.

One of the main motivations for this research was the surprisingly poor results Williams and Ellis got when they applied their formulation to the prediction of the precession of the perihelia of Mercury. Their best estimate was 9.1×10^{-4} radians per orbit. This is three orders of magnitude above the correct result of 5.0×10^{-7} radians per orbit. Their method was to directly measure the precession after one orbit. This method is unlikely to ever give accurate results primarily because of the sheer bulk of calculations required. If a precession of the order of 10^{-7} radians per orbit is to be accurately resolved then the number of angular sub-divisions should be at least of order $N_\theta = \pi/\Delta\theta \approx 10^7$. As the orbit is to be computed by a numerical integration it follows that at least 10^7 time steps will be required. It is highly unlikely that the numerical integration will retain sufficient accuracy after this many time steps that the precession could be resolved above the accumulated round-off and other errors. A further complication is that at the perihelia $dr/d\theta = 0$. Thus near the perihelia the radius varies as $(\Delta\theta)^2$ which in turn suggests that $(\Delta r)/r \approx 10^{-14}$. This requires $N_r \approx 10^{+14}$ which is far too many radial sub-divisions. Clearly a better approach is required.

The main idea to be developed here is that if a reasonable functional form of the path of Mercury can be proposed then a least squares analysis can be used to determine the parameters of the function. This then enables a trivial calculation for the precession. The main question must then be : What is a reasonable functional form for the path? As this is meant to be a numerical procedure applicable in other situations it is not acceptable to use an analytic solution of the geodesic equations. However the physics of the problem tells us that the motion must be very nearly circular. Thus it seem sensible to propose a functional form for $r(\theta)$ that is very nearly circular. Such a function is

$$r = \frac{A}{1 + B\cos(C\theta)}$$

for some constants A, B and C provided $B \ll 1$ and $C \approx 1$. By writing $C = 1 - D$ this may also be written as

$$r = \frac{A}{1 + B\cos\theta - BD\theta\sin\theta}$$

having discarded higher order terms in B and D. This leads to

$$\frac{1}{r} = A' + B'\cos\theta + C'\theta\sin\theta$$

where $A' = 1/A, B' = B/A$ and $C' = -BD/A$. This function is linear in A', B' and C' and thus a least squares analysis is readily employed.

Given A', B' and C' the perihelion shift $\Delta\phi$ can be calculated from

$$\Delta\phi = \frac{2\pi(1-C)}{C} = \frac{-2\pi C'}{B'+C'}$$

The results for Mercury are listed in table (6). The best result is 5.01×10^{-7} which is only 0.2% in error. The table is incomplete because, as outlined in the previous sections, the best results can be expected for tilings for which $u_i \approx v_i$.

The above method need only be applied in situations where the orbits are expected to be very nearly circular and the precession to be very small. In other cases such as for a binary-pulsar, the precession can **not** be assumed to be small and this method would not be appropriate. In which case it may be better and perhaps quite practical to revert to the original Williams and Ellis method.

6. Discussion.

The results of the previous section seem to validate the procedure followed in section 4 in obtaining the modified Regge equations. However, there still remains some questions about how that procedure may be adapted to a generic spacetime. For example consider the form of the interpolation used in the modified Regge equations (4.6). The interpolation coefficients, $\cos\tilde{\theta}, \sin\tilde{\theta}$, arose naturally as a consequence of the spherical symmetry of the Schwarzschild spacetime. Such symmetries do not exist in a generic spacetime. In such cases it will be necessary to choose interpolation coefficients that vary continuously across the tile and its immediate neighbours. A further condition is that the modified Regge equations take on their original unmodified form at the barycentre (for example) of each tile. These conditions can easily be satisfied using a piecewise continuous linear interpolation.

The nature of the spacetime map may also require some attention. In the current example it was used as a means of comparing paths from different spacetimes. In the generic Regge spacetime the map would take on a slightly different role. The basic data would be the discrete Regge spacetime out of which we would like to extract smooth quantities, such as the metric and geodesics. The spacetime map would play a role in this semi-local smoothing process. Indeed in the example described in section 4 one could have started without any knowledge of the metric coefficients of the Schwarzschild spacetime and after the interpolation process arrived at approximations for those coefficients, eg. the cubic approximation $\tilde{n}(u, v)$ to $n(r)$.

There is another aspect of the spacetime map that needs investigation. In obtaining the equation (4.5c,d) it was assumed that $\tilde{n}$ was constant in each tile. Then, by arguing in favour of having continuous coefficients for that equation, it was concluded that $\tilde{n}$ should vary smoothly across the tile boundaries. For this purpose a simple cubic interpolation (4.9) was chosen. Suppose now that this interpolated $\tilde{n}$ had been used from the beginning. The result would be an equation similar to (4.5c,d) but with the addition of terms involving $\tilde{n}'$.

There are two related reasons why the early introduction of an interpolated $\tilde{n}$ should not be used. By replacing $\tilde{n}$ throughout equation (4.5) with the interpolated $\tilde{n}$ one introduces an $O(\Delta)$ variation in the spacetime map. Whereas the contrary procedure would lead to an

$O(1)$ variation in equations (4.5c,d). It seems far better to pursue an interpolation process that has minimal effect on the equations. The second and more compelling reason is that the extra $\tilde{n}'$ terms in the Regge equations have no corresponding terms in the Schwarzschild equations. Thus it would seem that the correct approach is to defer any interpolation until (the first form of) the spacetime map has been obtained. Certainly this seems to work but the arguments just presented are rather weak. Clearly more work is required on this matter.

The final point to note is that the modified Regge equations does not describe a geodesic in the Regge spacetime. They describe a related path which differs (locally) from the Regge geodesic by terms of order $O(\Delta^2)$. The need to use the modified Regge equations arose from the poor rate of convergence of the true Regge geodesics to the known Schwarzschild geodesics.

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