The metric

$$g_{ab}\left(x\right)=\left(g_{ab}-\frac{1}{3}\,x^{c}x^{d}R_{acbd}-\frac{1}{6}\,x^{c}x^{d}x^{e}\nabla_{c}R_{adbe}+\frac{2}{45}\,x^{c}x^{d}x^{e}x^{f}R_{acdg}R_{befg}-\frac{1}{20}\,x^{c}x^{d}x^{e}x^{f}\nabla_{cd}R_{aebf}+\frac{1}{45}\,x^{c}x^{d}x^{e}x^{f}x^{g}R_{acdh}\nabla_{e}R_{bfgh}+\frac{1}{45}\,x^{c}x^{d}x^{e}x^{f}x^{g}R_{bcdh}\nabla_{e}R_{afgh}-\frac{1}{90}\,x^{c}x^{d}x^{e}x^{f}x^{g}\nabla_{cde}R_{afbg}\right)+\mathcal{O}\left(\epsilon^{6}\right)$$

The inverse metric

$$g^{ab}\left(x\right)=\left(g^{ab}+\frac{1}{3}\,x^{c}x^{d}R^{a}{}_{c}{}^{b}{}_{d}+\frac{1}{6}\,x^{c}x^{d}x^{e}\nabla_{c}R^{a}{}_{d}{}^{b}{}_{e}+\frac{1}{15}\,x^{c}x^{d}x^{e}x^{f}R^{a}{}_{cdg}R^{b}{}_{efg}+\frac{1}{20}\,x^{c}x^{d}x^{e}x^{f}\nabla_{cd}R^{a}{}_{e}{}^{b}{}_{f}+\frac{1}{30}\,x^{c}x^{d}x^{e}x^{f}x^{g}R^{a}{}_{cdh}\nabla_{e}R^{b}{}_{fgh}+\frac{1}{30}\,x^{c}x^{d}x^{e}x^{f}x^{g}R^{b}{}_{cdh}\nabla_{e}R^{a}{}_{fgh}+\frac{1}{90}\,x^{c}x^{d}x^{e}x^{f}x^{g}\nabla_{cde}R^{a}{}_{f}{}^{b}{}_{g}\right)+\mathcal{O}\left(\epsilon^{6}\right)$$

The connection

Here we use $\gamma^{a}_{bc\underline{d}}$ as coefficients in the Taylor series expansion of the connection $\Gamma^{a}_{bc}(x)$ around $x=0$. The symbols $\Gamma^{a}_{bc\underline{d}}$ are reserved for the *generalised connections*.

$$\Gamma^{a}_{bc}(x)=\gamma^{a}_{bcd}\,x^{d}+\gamma^{a}_{bcde}\,x^{d}x^{e}+\gamma^{a}_{bcdef}\,x^{d}x^{e}x^{f}+\gamma^{a}_{bcdefg}\,x^{d}x^{e}x^{f}x^{g}+\mathcal{O}\left(\epsilon^{6}\right)$$

Here we also introduce the notation $(abc)(...)$ to denote symmetrisation over the specified indices. I find that this is easier to write than trying to wrap the indices in brackets.

$$\gamma^{a}_{(bc)d}=\frac{2}{3}\,R^{a}{}_{bdc}$$

$$\gamma^{a}_{(bc)de}=\frac{1}{12}\,\nabla^{a}R_{dbec}+\frac{1}{6}\,\nabla_{b}R^{a}{}_{dec}+\frac{1}{3}\,\nabla_{d}R^{a}{}_{bec}$$

$$\gamma^{a}_{(bc)def} = \frac{1}{20}\,\nabla^{a}{}_{d}R_{ebfc} + \frac{1}{10}\,\nabla_{db}R^{a}{}_{efc} + \frac{1}{10}\,\nabla_{de}R^{a}{}_{bfc} - \frac{4}{45}\,R^{a}{}_{bdg}R_{ecfg} - \frac{2}{45}\,R^{a}{}_{dbg}R_{ecfg} + \frac{8}{45}\,R^{a}{}_{deg}R_{fbcg} - \frac{2}{45}\,R^{a}{}_{gdb}R_{ecfg}$$

$$\begin{aligned}\gamma^{a}_{(bc)defg} = {} & \frac{1}{60}\,\nabla^{a}{}_{de}R_{fbgc} + \frac{1}{30}\,\nabla_{deb}R^{a}{}_{fgc} + \frac{1}{45}\,\nabla_{def}R^{a}{}_{bgc} - \frac{1}{45}\,\nabla^{a}R_{fbgh}R_{dceh} - \frac{1}{45}\,\nabla_{b}R^{a}{}_{fgh}R_{dceh} + \frac{1}{30}\,\nabla_{b}R_{fcgh}R^{a}{}_{deh} + \frac{4}{45}\,\nabla_{e}R^{a}{}_{fgh}R_{dbch} - \frac{2}{45}\,\nabla_{e}R_{fbgh}R^{a}{}_{cdh}\\ & - \frac{1}{45}\,\nabla_{e}R_{fbgh}R^{a}{}_{dch} - \frac{1}{45}\,\nabla_{e}R_{fbgh}R^{a}{}_{hdc} - \frac{2}{45}\,\nabla_{f}R^{a}{}_{bgh}R_{dceh} - \frac{1}{45}\,\nabla_{f}R^{a}{}_{gbh}R_{dceh} - \frac{1}{45}\,\nabla_{f}R^{a}{}_{hgb}R_{dceh} + \frac{4}{45}\,\nabla_{f}R_{gbch}R^{a}{}_{deh} - \frac{1}{36}\,\nabla_{h}R_{fbgc}R^{a}{}_{deh}\end{aligned}$$

Partial derivatives of the Riemann curvature tensor.

The first four partial derivatives of the Riemann tensor when expressed in terms of the Riemann tensor and its covariant derivatives are

$$R^{u}{}_{(bc\dot{v},a)} = \nabla_{a}R^{u}{}_{bcv}$$

$$R^{u}{}_{(cd\dot{v},ab)} = \nabla_{ab}R^{u}{}_{cdv}$$

$$2R^{u}{}_{(de\dot{v},abc)} = 2\,\nabla_{abc}R^{u}{}_{dev} - R_{vabf}\nabla_{c}R^{u}{}_{def} + R^{u}{}_{abf}\nabla_{c}R_{vdef}$$

$$5R^{u}{}_{(ef\dot{v},abcd)} = 5\,\nabla_{abcd}R^{u}{}_{efv} - 7\,R_{vabg}\nabla_{cd}R^{u}{}_{efg} + 7\,R^{u}{}_{abg}\nabla_{cd}R_{vefg}$$

Generalised connections in generic coordinates

The generalised connectiosn in generic coordinates (i.e. prior to adapting to Riemann normal coordinates) are

$$\Gamma^{a}_{bc}(x) = \Gamma^{a}{}_{bc}$$

$$\Gamma^{a}_{bcd}(x) = \partial_{b}\Gamma^{a}{}_{cd} - 2\,\Gamma^{a}{}_{be}\Gamma^{e}{}_{cd}$$

$$\Gamma^{a}_{bcde}(x) = -\Gamma^{f}{}_{bc}\partial_{f}\Gamma^{a}{}_{de} - 4\,\Gamma^{f}{}_{bc}\partial_{d}\Gamma^{a}{}_{ef} + \partial_{bc}\Gamma^{a}{}_{de} + 2\,\Gamma^{a}{}_{fg}\Gamma^{f}{}_{bc}\Gamma^{g}{}_{de} + 4\,\Gamma^{a}{}_{bf}\Gamma^{f}{}_{cg}\Gamma^{g}{}_{de} - 2\,\Gamma^{a}{}_{bf}\partial_{c}\Gamma^{f}{}_{de}$$

$$\begin{aligned}\Gamma^{a}_{bcdef}(x) = {} & 2\,\Gamma^{g}{}_{bc}\Gamma^{h}{}_{dg}\partial_{h}\Gamma^{a}{}_{ef} + 6\,\Gamma^{g}{}_{bc}\Gamma^{h}{}_{de}\partial_{g}\Gamma^{a}{}_{fh} - \partial_{g}\Gamma^{a}{}_{bc}\partial_{d}\Gamma^{g}{}_{ef} - 3\,\Gamma^{g}{}_{bc}\partial_{dg}\Gamma^{a}{}_{ef} + 12\,\Gamma^{g}{}_{bc}\Gamma^{h}{}_{dg}\partial_{e}\Gamma^{a}{}_{fh} + 6\,\Gamma^{g}{}_{bc}\Gamma^{h}{}_{de}\partial_{f}\Gamma^{a}{}_{gh} \\ & - 6\,\partial_{b}\Gamma^{a}{}_{cg}\partial_{d}\Gamma^{g}{}_{ef} - 6\,\Gamma^{g}{}_{bc}\partial_{de}\Gamma^{a}{}_{fg} + \partial_{bcd}\Gamma^{a}{}_{ef} - 12\,\Gamma^{a}{}_{gh}\Gamma^{g}{}_{bc}\Gamma^{h}{}_{di}\Gamma^{i}{}_{ef} + 6\,\Gamma^{a}{}_{gh}\Gamma^{g}{}_{bc}\partial_{d}\Gamma^{h}{}_{ef} - 4\,\Gamma^{a}{}_{bg}\Gamma^{g}{}_{hi}\Gamma^{h}{}_{cd}\Gamma^{i}{}_{ef} \\ & - 8\,\Gamma^{a}{}_{bg}\Gamma^{g}{}_{ch}\Gamma^{h}{}_{di}\Gamma^{i}{}_{ef} + 8\,\Gamma^{a}{}_{bg}\Gamma^{h}{}_{cd}\partial_{e}\Gamma^{g}{}_{fh} + 4\,\Gamma^{a}{}_{bg}\Gamma^{g}{}_{ch}\partial_{d}\Gamma^{h}{}_{ef} + 2\,\Gamma^{a}{}_{bg}\Gamma^{h}{}_{cd}\partial_{h}\Gamma^{g}{}_{ef} - 2\,\Gamma^{a}{}_{bg}\partial_{cd}\Gamma^{g}{}_{ef}\end{aligned}$$

$$\begin{aligned}\Gamma^{a}_{bcdefg}(x) = {} & -4\,\Gamma^{h}{}_{bc}\Gamma^{i}{}_{dh}\Gamma^{j}{}_{ei}\partial_{j}\Gamma^{a}{}_{fg} - 2\,\Gamma^{h}{}_{bc}\Gamma^{i}{}_{de}\Gamma^{j}{}_{hi}\partial_{j}\Gamma^{a}{}_{fg} - 16\,\Gamma^{h}{}_{bc}\Gamma^{i}{}_{de}\Gamma^{j}{}_{fh}\partial_{j}\Gamma^{a}{}_{gi} + 2\,\Gamma^{h}{}_{bi}\partial_{h}\Gamma^{a}{}_{cd}\partial_{e}\Gamma^{i}{}_{fg} + 4\,\Gamma^{h}{}_{bc}\partial_{i}\Gamma^{a}{}_{de}\partial_{f}\Gamma^{i}{}_{gh} + 8\,\Gamma^{h}{}_{bc}\Gamma^{i}{}_{dh}\partial_{ei}\Gamma^{a}{}_{fg} \\ & - 24\,\Gamma^{h}{}_{bc}\Gamma^{i}{}_{de}\Gamma^{j}{}_{fh}\partial_{i}\Gamma^{a}{}_{gj} - 12\,\Gamma^{h}{}_{bc}\Gamma^{i}{}_{de}\Gamma^{j}{}_{fg}\partial_{h}\Gamma^{a}{}_{ij} + 8\,\Gamma^{h}{}_{bc}\partial_{i}\Gamma^{a}{}_{dh}\partial_{e}\Gamma^{i}{}_{fg} + 12\,\Gamma^{h}{}_{bc}\partial_{h}\Gamma^{a}{}_{di}\partial_{e}\Gamma^{i}{}_{fg} + 24\,\Gamma^{h}{}_{bc}\Gamma^{i}{}_{de}\partial_{fh}\Gamma^{a}{}_{gi} + \Gamma^{h}{}_{bc}\partial_{i}\Gamma^{a}{}_{de}\partial_{h}\Gamma^{i}{}_{fg} \\ & - 4\,\partial_{b}\Gamma^{h}{}_{cd}\partial_{eh}\Gamma^{a}{}_{fg} - \partial_{h}\Gamma^{a}{}_{bc}\partial_{de}\Gamma^{h}{}_{fg} + 3\,\Gamma^{h}{}_{bc}\Gamma^{i}{}_{de}\partial_{hi}\Gamma^{a}{}_{fg} - 6\,\Gamma^{h}{}_{bc}\partial_{deh}\Gamma^{a}{}_{fg} - 32\,\Gamma^{h}{}_{bc}\Gamma^{i}{}_{dh}\Gamma^{j}{}_{ei}\partial_{f}\Gamma^{a}{}_{gj} - 16\,\Gamma^{h}{}_{bc}\Gamma^{i}{}_{de}\Gamma^{j}{}_{hi}\partial_{f}\Gamma^{a}{}_{gj} - 48\,\Gamma^{h}{}_{bc}\Gamma^{i}{}_{de}\Gamma^{j}{}_{fh}\partial_{g}\Gamma^{a}{}_{ij} \\ & + 16\,\Gamma^{h}{}_{bi}\partial_{c}\Gamma^{a}{}_{dh}\partial_{e}\Gamma^{i}{}_{fg} + 32\,\Gamma^{h}{}_{bc}\partial_{d}\Gamma^{a}{}_{ei}\partial_{f}\Gamma^{i}{}_{gh} + 24\,\Gamma^{h}{}_{bc}\Gamma^{i}{}_{dh}\partial_{ef}\Gamma^{a}{}_{gi} + 24\,\Gamma^{h}{}_{bc}\partial_{d}\Gamma^{a}{}_{hi}\partial_{e}\Gamma^{i}{}_{fg} + 12\,\Gamma^{h}{}_{bc}\Gamma^{i}{}_{de}\partial_{fg}\Gamma^{a}{}_{hi} + 8\,\Gamma^{h}{}_{bc}\partial_{d}\Gamma^{a}{}_{ei}\partial_{h}\Gamma^{i}{}_{fg} \\ & - 12\,\partial_{b}\Gamma^{h}{}_{cd}\partial_{ef}\Gamma^{a}{}_{gh} - 8\,\partial_{b}\Gamma^{a}{}_{ch}\partial_{de}\Gamma^{h}{}_{fg} - 8\,\Gamma^{h}{}_{bc}\partial_{def}\Gamma^{a}{}_{gh} + \partial_{bcde}\Gamma^{a}{}_{fg} + 24\,\Gamma^{a}{}_{hi}\Gamma^{h}{}_{bj}\Gamma^{i}{}_{ck}\Gamma^{j}{}_{de}\Gamma^{k}{}_{fg} + 16\,\Gamma^{a}{}_{hi}\Gamma^{h}{}_{bc}\Gamma^{i}{}_{jk}\Gamma^{j}{}_{de}\Gamma^{k}{}_{fg} + 32\,\Gamma^{a}{}_{hi}\Gamma^{h}{}_{bc}\Gamma^{i}{}_{dj}\Gamma^{j}{}_{ek}\Gamma^{k}{}_{fg} \\ & - 24\,\Gamma^{a}{}_{hi}\Gamma^{h}{}_{bj}\Gamma^{j}{}_{cd}\partial_{e}\Gamma^{i}{}_{fg} - 32\,\Gamma^{a}{}_{hi}\Gamma^{h}{}_{bc}\Gamma^{j}{}_{de}\partial_{f}\Gamma^{i}{}_{gj} - 16\,\Gamma^{a}{}_{hi}\Gamma^{h}{}_{bc}\Gamma^{i}{}_{dj}\partial_{e}\Gamma^{j}{}_{fg} - 8\,\Gamma^{a}{}_{hi}\Gamma^{h}{}_{bc}\Gamma^{j}{}_{de}\partial_{j}\Gamma^{i}{}_{fg} + 6\,\Gamma^{a}{}_{hi}\partial_{b}\Gamma^{h}{}_{cd}\partial_{e}\Gamma^{i}{}_{fg} + 8\,\Gamma^{a}{}_{hi}\Gamma^{h}{}_{bc}\partial_{de}\Gamma^{i}{}_{fg} \\ & + 24\,\Gamma^{a}{}_{bh}\Gamma^{h}{}_{ij}\Gamma^{i}{}_{cd}\Gamma^{j}{}_{ek}\Gamma^{k}{}_{fg} - 12\,\Gamma^{a}{}_{bh}\Gamma^{i}{}_{cd}\Gamma^{j}{}_{ef}\partial_{g}\Gamma^{h}{}_{ij} - 12\,\Gamma^{a}{}_{bh}\Gamma^{h}{}_{ij}\Gamma^{i}{}_{cd}\partial_{e}\Gamma^{j}{}_{fg} + 8\,\Gamma^{a}{}_{bh}\Gamma^{h}{}_{ci}\Gamma^{i}{}_{jk}\Gamma^{j}{}_{de}\Gamma^{k}{}_{fg} + 16\,\Gamma^{a}{}_{bh}\Gamma^{h}{}_{ci}\Gamma^{i}{}_{dj}\Gamma^{j}{}_{ek}\Gamma^{k}{}_{fg} \\ & - 24\,\Gamma^{a}{}_{bh}\Gamma^{i}{}_{cd}\Gamma^{j}{}_{ei}\partial_{f}\Gamma^{h}{}_{gj} - 16\,\Gamma^{a}{}_{bh}\Gamma^{h}{}_{ci}\Gamma^{j}{}_{de}\partial_{f}\Gamma^{i}{}_{gj} - 8\,\Gamma^{a}{}_{bh}\Gamma^{h}{}_{ci}\Gamma^{i}{}_{dj}\partial_{e}\Gamma^{j}{}_{fg} - 12\,\Gamma^{a}{}_{bh}\Gamma^{i}{}_{cd}\Gamma^{j}{}_{ef}\partial_{i}\Gamma^{h}{}_{gj} + 12\,\Gamma^{a}{}_{bh}\partial_{c}\Gamma^{h}{}_{di}\partial_{e}\Gamma^{i}{}_{fg} + 12\,\Gamma^{a}{}_{bh}\Gamma^{i}{}_{cd}\partial_{ef}\Gamma^{h}{}_{gi} \\ & - 4\,\Gamma^{a}{}_{bh}\Gamma^{h}{}_{ci}\Gamma^{j}{}_{de}\partial_{j}\Gamma^{i}{}_{fg} + 4\,\Gamma^{a}{}_{bh}\Gamma^{h}{}_{ci}\partial_{de}\Gamma^{i}{}_{fg} - 4\,\Gamma^{a}{}_{bh}\Gamma^{i}{}_{cd}\Gamma^{j}{}_{ei}\partial_{j}\Gamma^{h}{}_{fg} + 2\,\Gamma^{a}{}_{bh}\partial_{c}\Gamma^{i}{}_{de}\partial_{i}\Gamma^{h}{}_{fg} + 6\,\Gamma^{a}{}_{bh}\Gamma^{i}{}_{cd}\partial_{ei}\Gamma^{h}{}_{fg} - 2\,\Gamma^{a}{}_{bh}\partial_{cde}\Gamma^{h}{}_{fg}\end{aligned}$$

Generalised connections in RNC coordinates

The generalised connections in Riemann normal coordinates.

$$\begin{aligned}\Gamma^{a}_{bc}(x) = {} & \frac{2}{3}\,x^{d}R^{a}{}_{bdc} + \frac{1}{6}\,x^{d}x^{e}\nabla_{b}R^{a}{}_{dec} + \frac{1}{3}\,x^{d}x^{e}\nabla_{d}R^{a}{}_{bec} + \frac{1}{12}\,x^{d}x^{e}\nabla^{a}R_{dbec} + \frac{8}{45}\,x^{d}x^{e}x^{f}R^{a}{}_{deg}R_{fbcg} - \frac{4}{45}\,x^{d}x^{e}x^{f}R^{a}{}_{bdg}R_{ecfg} - \frac{2}{45}\,x^{d}x^{e}x^{f}R^{a}{}_{dbg}R_{ecfg} \\
& + \frac{1}{10}\,x^{d}x^{e}x^{f}\nabla_{db}R^{a}{}_{efc} + \frac{1}{10}\,x^{d}x^{e}x^{f}\nabla_{de}R^{a}{}_{bfc} - \frac{2}{45}\,x^{d}x^{e}x^{f}R^{a}{}_{gdb}R_{ecfg} + \frac{1}{20}\,x^{d}x^{e}x^{f}\nabla^{a}{}_{d}R_{ebfc} + \frac{4}{45}\,x^{d}x^{e}x^{f}x^{g}R_{dbch}\nabla_{e}R^{a}{}_{fgh} \\
& + \frac{1}{30}\,x^{d}x^{e}x^{f}x^{g}R^{a}{}_{deh}\nabla_{b}R_{fcgh} + \frac{4}{45}\,x^{d}x^{e}x^{f}x^{g}R^{a}{}_{deh}\nabla_{f}R_{gbch} - \frac{2}{45}\,x^{d}x^{e}x^{f}x^{g}R^{a}{}_{bdh}\nabla_{e}R_{fcgh} - \frac{1}{45}\,x^{d}x^{e}x^{f}x^{g}R^{a}{}_{dbh}\nabla_{e}R_{fcgh} - \frac{1}{45}\,x^{d}x^{e}x^{f}x^{g}R_{dbeh}\nabla_{c}R^{a}{}_{fgh} \\
& - \frac{2}{45}\,x^{d}x^{e}x^{f}x^{g}R_{dbeh}\nabla_{f}R^{a}{}_{cgh} - \frac{1}{45}\,x^{d}x^{e}x^{f}x^{g}R_{dbeh}\nabla_{f}R^{a}{}_{gch} + \frac{1}{30}\,x^{d}x^{e}x^{f}x^{g}\nabla_{deb}R^{a}{}_{fgc} + \frac{1}{45}\,x^{d}x^{e}x^{f}x^{g}\nabla_{def}R^{a}{}_{bgc} - \frac{1}{36}\,x^{d}x^{e}x^{f}x^{g}R^{a}{}_{deh}\nabla_{h}R_{fbgc} \\
& - \frac{1}{45}\,x^{d}x^{e}x^{f}x^{g}R^{a}{}_{hdb}\nabla_{e}R_{fcgh} - \frac{1}{45}\,x^{d}x^{e}x^{f}x^{g}R_{dbeh}\nabla^{a}R_{fcgh} - \frac{1}{45}\,x^{d}x^{e}x^{f}x^{g}R_{dbeh}\nabla_{f}R^{a}{}_{hgc} + \frac{1}{60}\,x^{d}x^{e}x^{f}x^{g}\nabla^{a}{}_{de}R_{fbgc} + \mathcal{O}\left(\epsilon^{6}\right)\end{aligned}$$

$$\begin{aligned}\Gamma^{a}_{bcd}(x) = {} & \frac{1}{2}\,x^{e}\nabla_{b}R^{a}{}_{ced} + \frac{2}{15}\,x^{e}x^{f}R^{a}{}_{ebg}R_{fcdg} + \frac{8}{15}\,x^{e}x^{f}R^{a}{}_{beg}R_{fcdg} - \frac{2}{15}\,x^{e}x^{f}R^{a}{}_{bcg}R_{edfg} + \frac{3}{10}\,x^{e}x^{f}\nabla_{eb}R^{a}{}_{cfd} + \frac{1}{10}\,x^{e}x^{f}\nabla_{bc}R^{a}{}_{efd} \\
& + \frac{2}{5}\,x^{e}x^{f}R^{a}{}_{geb}R_{fcdg} + \frac{1}{20}\,x^{e}x^{f}\nabla^{a}{}_{b}R_{ecfd} + \frac{1}{9}\,x^{e}x^{f}x^{g}R_{ebch}\nabla_{d}R^{a}{}_{fgh} + \frac{1}{9}\,x^{e}x^{f}x^{g}R_{ebch}\nabla_{h}R^{a}{}_{fgd} + \frac{2}{9}\,x^{e}x^{f}x^{g}R_{ebch}\nabla_{f}R^{a}{}_{dgh} \\
& + \frac{2}{9}\,x^{e}x^{f}x^{g}R_{ebch}\nabla_{f}R^{a}{}_{hgd} + \frac{1}{9}\,x^{e}x^{f}x^{g}R_{ebch}\nabla^{a}R_{fdgh} + \frac{1}{9}\,x^{e}x^{f}x^{g}R^{a}{}_{beh}\nabla_{c}R_{fdgh} + \frac{2}{9}\,x^{e}x^{f}x^{g}R^{a}{}_{beh}\nabla_{f}R_{gcdh} \\
& - \frac{1}{18}\,x^{e}x^{f}x^{g}R^{a}{}_{beh}\nabla_{h}R_{fcgd} + \frac{1}{9}\,x^{e}x^{f}x^{g}R^{a}{}_{heb}\nabla_{c}R_{fdgh} + \frac{2}{9}\,x^{e}x^{f}x^{g}R^{a}{}_{heb}\nabla_{f}R_{gcdh} - \frac{1}{18}\,x^{e}x^{f}x^{g}R^{a}{}_{heb}\nabla_{h}R_{fcgd} + \mathcal{O}\left(\epsilon^{6}\right)\end{aligned}$$

$$\begin{aligned}\Gamma^{a}_{bcde}(x) = {} & \frac{4}{9}\,x^{f}R^{a}{}_{bcg}R_{fdeg} + \frac{1}{9}\,x^{f}x^{g}R^{a}{}_{bch}\nabla_{d}R_{fegh} + \frac{2}{9}\,x^{f}x^{g}R^{a}{}_{bch}\nabla_{f}R_{gdeh} - \frac{1}{18}\,x^{f}x^{g}R^{a}{}_{bch}\nabla_{h}R_{fdge} + \frac{1}{9}\,x^{f}x^{g}R_{fbch}\nabla_{d}R^{a}{}_{geh} + \frac{5}{9}\,x^{f}x^{g}R_{fbch}\nabla_{d}R^{a}{}_{hge} \\
& + \frac{2}{9}\,x^{f}x^{g}R_{fbch}\nabla_{g}R^{a}{}_{deh} + \frac{4}{9}\,x^{f}x^{g}R_{fbch}\nabla_{h}R^{a}{}_{dge} + \frac{1}{9}\,x^{f}x^{g}R_{fbch}\nabla^{a}R_{gdeh} + \frac{2}{3}\,x^{f}x^{g}R_{fbch}\nabla_{d}R^{a}{}_{egh} + \frac{1}{3}\,x^{f}x^{g}R^{a}{}_{bfh}\nabla_{c}R_{gdeh} + \frac{1}{3}\,x^{f}x^{g}R^{a}{}_{hfb}\nabla_{c}R_{gdeh} + \mathcal{O}\left(\epsilon^{6}\right)\end{aligned}$$

$$\Gamma^{a}_{bcdef}(x) = \frac{2}{3}\,x^{g}R^{a}{}_{bch}\nabla_{d}R_{gefh} + x^{g}R_{gbch}\nabla_{d}R^{a}{}_{efh} + \mathcal{O}\left(\epsilon^{6}\right)$$

$$\Gamma^{a}_{bcdefg}(x) = 0$$

Transformation from generic to RNC coordinates

The coordinate transformation from generic coordinates x^a to Riemann normal coordinates y^a.

$$
\begin{aligned}
y_5^a = \Bigg(& \Delta x^{a} + \frac{1}{2}\,\Delta x^{b}\Delta x^{c}\Gamma^{a}{}_{bc} + \Delta x^{b}\Delta x^{c}\Delta x^{d}\left(\frac{1}{6}\,\Gamma^{a}{}_{be}\Gamma^{e}{}_{cd} + \frac{1}{6}\,\partial_b\Gamma^{a}{}_{cd}\right) + \Delta x^{b}\Delta x^{c}\Delta x^{d}\Delta x^{e}\left(\frac{1}{12}\,\Gamma^{a}{}_{bf}\partial_c\Gamma^{f}{}_{de} + \frac{1}{24}\,\Gamma^{a}{}_{fg}\Gamma^{f}{}_{bc}\Gamma^{g}{}_{de} + \frac{1}{24}\,\Gamma^{f}{}_{bc}\partial_f\Gamma^{a}{}_{de} + \frac{1}{24}\,\partial_{bc}\Gamma^{a}{}_{de}\right) \\
& + \Delta x^{b}\Delta x^{c}\Delta x^{d}\Delta x^{e}\Delta x^{f}\Bigg(-\frac{1}{90}\,\Gamma^{a}{}_{bg}\Gamma^{g}{}_{ch}\Gamma^{h}{}_{di}\Gamma^{i}{}_{ef} + \frac{1}{180}\,\Gamma^{a}{}_{bg}\Gamma^{g}{}_{ch}\partial_d\Gamma^{h}{}_{ef} + \frac{1}{120}\,\Gamma^{a}{}_{bg}\Gamma^{g}{}_{hi}\Gamma^{h}{}_{cd}\Gamma^{i}{}_{ef} + \frac{1}{60}\,\Gamma^{a}{}_{bg}\Gamma^{h}{}_{cd}\partial_h\Gamma^{g}{}_{ef} - \frac{1}{60}\,\Gamma^{a}{}_{bg}\Gamma^{h}{}_{cd}\partial_e\Gamma^{g}{}_{fh} \\
& + \frac{1}{40}\,\Gamma^{a}{}_{bg}\partial_{cd}\Gamma^{g}{}_{ef} + \frac{1}{90}\,\Gamma^{a}{}_{gh}\Gamma^{g}{}_{bc}\Gamma^{h}{}_{di}\Gamma^{i}{}_{ef} + \frac{13}{360}\,\Gamma^{a}{}_{gh}\Gamma^{g}{}_{bc}\partial_d\Gamma^{h}{}_{ef} + \frac{1}{360}\,\Gamma^{g}{}_{bc}\Gamma^{h}{}_{dg}\partial_h\Gamma^{a}{}_{ef} - \frac{1}{90}\,\Gamma^{g}{}_{bc}\Gamma^{h}{}_{dg}\partial_e\Gamma^{a}{}_{fh} + \frac{7}{360}\,\partial_g\Gamma^{a}{}_{bc}\partial_d\Gamma^{g}{}_{ef} + \frac{1}{180}\,\partial_b\Gamma^{a}{}_{cg}\partial_d\Gamma^{g}{}_{ef} \\
& + \frac{1}{120}\,\Gamma^{g}{}_{bc}\Gamma^{h}{}_{de}\partial_g\Gamma^{a}{}_{fh} + \frac{1}{120}\,\Gamma^{g}{}_{bc}\Gamma^{h}{}_{de}\partial_f\Gamma^{a}{}_{gh} + \frac{1}{60}\,\Gamma^{g}{}_{bc}\partial_{dg}\Gamma^{a}{}_{ef} - \frac{1}{120}\,\Gamma^{g}{}_{bc}\partial_{de}\Gamma^{a}{}_{fg} + \frac{1}{120}\,\partial_{bcd}\Gamma^{a}{}_{ef}\Bigg)\Bigg)
\end{aligned}
$$

The geodesic IVP

The Taylor series solution to the geodesic equation

$$0 = \frac{d^2x^a}{ds^2} + \Gamma^a_{bc}(x)\frac{dx^b}{ds}\frac{dx^c}{ds}$$

subject to the initial conditions $x^a(s) = x^a$ and $dx^a(s)/ds = \dot{x}^a$ at $s = 0$, is

$$\begin{aligned}
x^a(s) = \Bigg(& x^a + s\dot{x}^a - \frac{1}{2} s^2 \dot{x}^b \dot{x}^c \Bigg(\frac{2}{3} x^d R^a{}_{bdc} + \frac{1}{6} x^d x^e \nabla_b R^a{}_{dec} + \frac{1}{3} x^d x^e \nabla_d R^a{}_{bec} + \frac{1}{12} x^d x^e \nabla^a R_{dbec} + \frac{8}{45} x^d x^e x^f R^a{}_{deg} R_{fbcg} - \frac{4}{45} x^d x^e x^f R^a{}_{bdg} R_{ecfg} \\
& - \frac{2}{45} x^d x^e x^f R^a{}_{dbg} R_{ecfg} + \frac{1}{10} x^d x^e x^f \nabla_{db} R^a{}_{efc} + \frac{1}{10} x^d x^e x^f \nabla_{de} R^a{}_{bfc} - \frac{2}{45} x^d x^e x^f R^a{}_{gdb} R_{ecfg} + \frac{1}{20} x^d x^e x^f \nabla^a{}_d R_{ebfc} + \frac{4}{45} x^d x^e x^f x^g R_{dbch} \nabla_e R^a{}_{fgh} \\
& + \frac{1}{30} x^d x^e x^f x^g R^a{}_{deh} \nabla_b R_{fcgh} + \frac{4}{45} x^d x^e x^f x^g R^a{}_{deh} \nabla_f R_{gbch} - \frac{2}{45} x^d x^e x^f x^g R^a{}_{bdh} \nabla_e R_{fcgh} - \frac{1}{45} x^d x^e x^f x^g R^a{}_{dbh} \nabla_e R_{fcgh} - \frac{1}{45} x^d x^e x^f x^g R_{dbeh} \nabla_c R^a{}_{fgh} \\
& - \frac{2}{45} x^d x^e x^f x^g R_{dbeh} \nabla_f R^a{}_{cgh} - \frac{1}{45} x^d x^e x^f x^g R_{dbeh} \nabla_f R^a{}_{gch} + \frac{1}{30} x^d x^e x^f x^g \nabla_{deb} R^a{}_{fgc} + \frac{1}{45} x^d x^e x^f x^g \nabla_{def} R^a{}_{bgc} - \frac{1}{36} x^d x^e x^f x^g R^a{}_{deh} \nabla_h R_{fbgc} \\
& - \frac{1}{45} x^d x^e x^f x^g R^a{}_{hdb} \nabla_e R_{fcgh} - \frac{1}{45} x^d x^e x^f x^g R_{dbeh} \nabla^a R_{fcgh} - \frac{1}{45} x^d x^e x^f x^g R_{dbeh} \nabla_f R^a{}_{hgc} + \frac{1}{60} x^d x^e x^f x^g \nabla^a{}_{de} R_{fbgc} \Bigg) - \frac{1}{6} s^3 \dot{x}^b \dot{x}^c \dot{x}^d \Bigg(\frac{1}{2} x^e \nabla_b R^a{}_{ced} \\
& + \frac{2}{15} x^e x^f R^a{}_{ebg} R_{fcdg} + \frac{8}{15} x^e x^f R^a{}_{beg} R_{fcdg} - \frac{2}{15} x^e x^f R^a{}_{bcg} R_{edfg} + \frac{3}{10} x^e x^f \nabla_{eb} R^a{}_{cfd} + \frac{1}{10} x^e x^f \nabla_{bc} R^a{}_{efd} + \frac{2}{5} x^e x^f R^a{}_{geb} R_{fcdg} + \frac{1}{20} x^e x^f \nabla^a{}_b R_{ecfd} \\
& + \frac{1}{9} x^e x^f x^g R_{ebch} \nabla_d R^a{}_{fgh} + \frac{1}{9} x^e x^f x^g R_{ebch} \nabla_h R^a{}_{fgd} + \frac{2}{9} x^e x^f x^g R_{ebch} \nabla_f R^a{}_{dgh} + \frac{2}{9} x^e x^f x^g R_{ebch} \nabla_f R^a{}_{hgd} + \frac{1}{9} x^e x^f x^g R_{ebch} \nabla^a R_{fdgh} + \frac{1}{9} x^e x^f x^g R^a{}_{beh} \nabla_c R_{fdgh} \\
& + \frac{2}{9} x^e x^f x^g R^a{}_{beh} \nabla_f R_{gcdh} - \frac{1}{18} x^e x^f x^g R^a{}_{beh} \nabla_h R_{fcgd} + \frac{1}{9} x^e x^f x^g R^a{}_{heb} \nabla_c R_{fdgh} + \frac{2}{9} x^e x^f x^g R^a{}_{heb} \nabla_f R_{gcdh} - \frac{1}{18} x^e x^f x^g R^a{}_{heb} \nabla_h R_{fcgd} \Bigg) \\
& - \frac{1}{24} s^4 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e \Bigg(\frac{4}{9} x^f R^a{}_{bcg} R_{fdeg} + \frac{1}{9} x^f x^g R^a{}_{bch} \nabla_d R_{fegh} + \frac{2}{9} x^f x^g R^a{}_{bch} \nabla_f R_{gdeh} - \frac{1}{18} x^f x^g R^a{}_{bch} \nabla_h R_{fdge} + \frac{1}{9} x^f x^g R_{fbch} \nabla_d R^a{}_{geh} + \frac{5}{9} x^f x^g R_{fbch} \nabla_d R^a{}_{hge} \\
& + \frac{2}{9} x^f x^g R_{fbch} \nabla_g R^a{}_{deh} + \frac{4}{9} x^f x^g R_{fbch} \nabla_h R^a{}_{dge} + \frac{1}{9} x^f x^g R_{fbch} \nabla^a R_{gdeh} + \frac{2}{3} x^f x^g R_{fbch} \nabla_d R^a{}_{egh} + \frac{1}{3} x^f x^g R^a{}_{bfh} \nabla_c R_{gdeh} + \frac{1}{3} x^f x^g R^a{}_{hfb} \nabla_c R_{gdeh} \Bigg) \\
& - \frac{1}{120} s^5 \dot{x}^b \dot{x}^c \dot{x}^d \dot{x}^e \dot{x}^f \Bigg(\frac{2}{3} x^g R^a{}_{bch} \nabla_d R_{gefh} + x^g R_{gbch} \nabla_d R^a{}_{efh} \Bigg) \Bigg) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}$$

The geodesic BVP

Consider a geodesic that connects two points P and Q with coordinates x^a and $x^a + \Delta x^a$. We choose a parameter s along the geodesic with $s = 0$ at P and $s = 1$ at Q. The solution $x(s)$ to the geodesic boundary value problem is

$$x^a(s) = x^a_0 + x^a_1 s + x^a_2 s^2 + x^a_3 s^3 + x^a_4 s^4 + x^a_5 s^5 + \mathcal{O}\left(\epsilon^6\right)$$

with

$$x^a_0 = x^a$$

$$\begin{aligned}
x^a_1 = \Bigg(&\Delta x^{\,a} + \Delta x^{\,b}\Delta x^{\,c}\bigg(\frac{1}{3}\,x^d {R^a}_{\,bdc} + \frac{1}{12}\,x^d x^e \nabla_b {R^a}_{\,dec} + \frac{1}{6}\,x^d x^e \nabla_d {R^a}_{\,bec} + \frac{1}{24}\,x^d x^e \nabla^a R_{dbec} + \frac{4}{45}\,x^d x^e x^f {R^a}_{\,dge} R_{gbfc} - \frac{2}{45}\,x^d x^e x^f {R^a}_{\,bgd} R_{gefc} - \frac{1}{45}\,x^d x^e x^f {R^a}_{\,dgb} R_{gefc} \\
&+ \frac{1}{20}\,x^d x^e x^f \nabla_{db} {R^a}_{\,efc} + \frac{1}{20}\,x^d x^e x^f \nabla_{de} {R^a}_{\,bfc} + \frac{1}{45}\,x^d x^e x^f {R^a}_{\,gdb} R_{gefc} + \frac{1}{40}\,x^d x^e x^f {\nabla^a}_{\,d} R_{ebfc} + \frac{2}{45}\,x^d x^e x^f x^g R_{hbdc} \nabla_e {R^a}_{\,fhg} + \frac{1}{60}\,x^d x^e x^f x^g {R^a}_{\,dhe} \nabla_b R_{hfgc} \\
&+ \frac{2}{45}\,x^d x^e x^f x^g {R^a}_{\,dhe} \nabla_f R_{hbgc} - \frac{1}{45}\,x^d x^e x^f x^g {R^a}_{\,bhd} \nabla_e R_{hfgc} - \frac{1}{90}\,x^d x^e x^f x^g {R^a}_{\,dhb} \nabla_e R_{hfgc} - \frac{1}{90}\,x^d x^e x^f x^g R_{hdeb} \nabla_c {R^a}_{\,fhg} - \frac{1}{45}\,x^d x^e x^f x^g R_{hdeb} \nabla_f {R^a}_{\,chg} \\
&- \frac{1}{90}\,x^d x^e x^f x^g R_{hdeb} \nabla_f {R^a}_{\,ghc} + \frac{1}{60}\,x^d x^e x^f x^g \nabla_{deb} {R^a}_{\,fgc} + \frac{1}{90}\,x^d x^e x^f x^g \nabla_{def} {R^a}_{\,bgc} + \frac{1}{72}\,x^d x^e x^f x^g {R^a}_{\,dhe} \nabla_h R_{fbgc} + \frac{1}{90}\,x^d x^e x^f x^g {R^a}_{\,hdb} \nabla_e R_{hfgc} \\
&- \frac{1}{90}\,x^d x^e x^f x^g R_{hdeb} \nabla^a R_{hfgc} + \frac{1}{90}\,x^d x^e x^f x^g R_{hdeb} \nabla_f {R^a}_{\,hgc} + \frac{1}{120}\,x^d x^e x^f x^g {\nabla^a}_{\,de} R_{fbgc}\bigg) + \Delta x^{\,b}\Delta x^{\,c}\Delta x^{\,d}\bigg(\frac{1}{12}\,x^e \nabla_b {R^a}_{\,ced} + \frac{1}{45}\,x^e x^f {R^a}_{\,egb} R_{gcfd} \\
&+ \frac{4}{45}\,x^e x^f {R^a}_{\,bge} R_{gcfd} - \frac{1}{45}\,x^e x^f {R^a}_{\,bgc} R_{gefd} + \frac{1}{20}\,x^e x^f \nabla_{eb} {R^a}_{\,cfd} + \frac{1}{60}\,x^e x^f \nabla_{bc} {R^a}_{\,efd} + \frac{7}{45}\,x^e x^f {R^a}_{\,geb} R_{gcfd} + \frac{1}{120}\,x^e x^f {\nabla^a}_{\,b} R_{ecfd} + \frac{1}{54}\,x^e x^f x^g R_{hbec} \nabla_d {R^a}_{\,fhg} \\
&+ \frac{1}{27}\,x^e x^f x^g R_{hbec} \nabla_h {R^a}_{\,fgd} + \frac{1}{27}\,x^e x^f x^g R_{hbec} \nabla_f {R^a}_{\,dhg} + \frac{2}{27}\,x^e x^f x^g R_{hbec} \nabla_f {R^a}_{\,hgd} - \frac{1}{108}\,x^e x^f x^g R_{hbec} \nabla^a R_{hfgd} + \frac{1}{54}\,x^e x^f x^g {R^a}_{\,bhe} \nabla_c R_{hfgd} \\
&+ \frac{1}{27}\,x^e x^f x^g {R^a}_{\,bhe} \nabla_f R_{hcgd} + \frac{1}{108}\,x^e x^f x^g {R^a}_{\,bhe} \nabla_h R_{fcgd} + \frac{1}{27}\,x^e x^f x^g {R^a}_{\,heb} \nabla_c R_{hfgd} + \frac{2}{27}\,x^e x^f x^g {R^a}_{\,heb} \nabla_f R_{hcgd} + \frac{1}{54}\,x^e x^f x^g {R^a}_{\,heb} \nabla_h R_{fcgd}\bigg) \\
&+ \Delta x^{\,b}\Delta x^{\,c}\Delta x^{\,d}\Delta x^{\,e}\bigg(\frac{1}{54}\,x^f {R^a}_{\,bgc} R_{gdfe} + \frac{1}{216}\,x^f x^g {R^a}_{\,bhc} \nabla_d R_{hfge} + \frac{1}{108}\,x^f x^g {R^a}_{\,bhc} \nabla_f R_{hdge} + \frac{1}{432}\,x^f x^g {R^a}_{\,bhc} \nabla_h R_{fdge} + \frac{1}{216}\,x^f x^g R_{hbfc} \nabla_d {R^a}_{\,ghe} \\
&- \frac{5}{216}\,x^f x^g R_{hbfc} \nabla_d {R^a}_{\,hge} + \frac{1}{108}\,x^f x^g R_{hbfc} \nabla_g {R^a}_{\,dhe} + \frac{7}{108}\,x^f x^g R_{hbfc} \nabla_h {R^a}_{\,dge} + \frac{1}{216}\,x^f x^g R_{hbfc} \nabla^a R_{hdge} + \frac{1}{36}\,x^f x^g R_{hbfc} \nabla_d {R^a}_{\,ehg} + \frac{1}{72}\,x^f x^g {R^a}_{\,bhf} \nabla_c R_{hdge} \\
&+ \frac{1}{24}\,x^f x^g {R^a}_{\,hfb} \nabla_c R_{hdge}\bigg) + \Delta x^{\,b}\Delta x^{\,c}\Delta x^{\,d}\Delta x^{\,e}\Delta x^{\,f}\left(\frac{1}{180}\,x^g {R^a}_{\,bhc} \nabla_d R_{hegf} + \frac{1}{120}\,x^g R_{hbgc} \nabla_d {R^a}_{\,ehf}\right)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
x_2^a = \Bigg(& \Delta x^{\,b} \Delta x^{\,c} \Bigg(-\frac{1}{3}\, x^d R^a{}_{bdc} - \frac{1}{12}\, x^d x^e \nabla_b R^a{}_{dec} - \frac{1}{6}\, x^d x^e \nabla_d R^a{}_{bec} - \frac{1}{24}\, x^d x^e \nabla^a R_{dbec} - \frac{4}{45}\, x^d x^e x^f R^a{}_{dge} R_{gbfc} + \frac{2}{45}\, x^d x^e x^f R^a{}_{bgd} R_{gefc} + \frac{1}{45}\, x^d x^e x^f R^a{}_{dgb} R_{gefc} \\
& - \frac{1}{20}\, x^d x^e x^f \nabla_{db} R^a{}_{efc} - \frac{1}{20}\, x^d x^e x^f \nabla_{de} R^a{}_{bfc} - \frac{1}{45}\, x^d x^e x^f R^a{}_{gdb} R_{gefc} - \frac{1}{40}\, x^d x^e x^f \nabla^a{}_d R_{ebfc} - \frac{2}{45}\, x^d x^e x^f x^g R_{hbdc} \nabla_e R^a{}_{fhg} - \frac{1}{60}\, x^d x^e x^f x^g R^a{}_{dhe} \nabla_b R_{hfgc} \\
& - \frac{2}{45}\, x^d x^e x^f x^g R^a{}_{dhe} \nabla_f R_{hbgc} + \frac{1}{45}\, x^d x^e x^f x^g R^a{}_{bhd} \nabla_e R_{hfgc} + \frac{1}{90}\, x^d x^e x^f x^g R^a{}_{dhb} \nabla_e R_{hfgc} + \frac{1}{90}\, x^d x^e x^f x^g R_{hdeb} \nabla_c R^a{}_{fhg} + \frac{1}{45}\, x^d x^e x^f x^g R_{hdeb} \nabla_f R^a{}_{chg} \\
& + \frac{1}{90}\, x^d x^e x^f x^g R_{hdeb} \nabla_f R^a{}_{ghc} - \frac{1}{60}\, x^d x^e x^f x^g \nabla_{deb} R^a{}_{fgc} - \frac{1}{90}\, x^d x^e x^f x^g \nabla_{def} R^a{}_{bgc} - \frac{1}{72}\, x^d x^e x^f x^g R^a{}_{dhe} \nabla_h R_{fbgc} - \frac{1}{90}\, x^d x^e x^f x^g R^a{}_{hdb} \nabla_e R_{hfgc} \\
& + \frac{1}{90}\, x^d x^e x^f x^g R_{hdeb} \nabla^a R_{hfgc} - \frac{1}{90}\, x^d x^e x^f x^g R_{hdeb} \nabla_f R^a{}_{hgc} - \frac{1}{120}\, x^d x^e x^f x^g \nabla^a{}_{de} R_{fbgc} \Bigg) + \Delta x^{\,b} \Delta x^{\,c} \Delta x^{\,d} \Bigg(-\frac{2}{9}\, x^e x^f R^a{}_{geb} R^g{}_{cfd} - \frac{1}{18}\, x^e x^f x^g R^a{}_{heb} \nabla_c R^h{}_{fgd} \\
& - \frac{1}{9}\, x^e x^f x^g R^a{}_{heb} \nabla_f R^h{}_{cgd} - \frac{1}{36}\, x^e x^f x^g R^a{}_{heb} \nabla^h R_{fcgd} - \frac{1}{18}\, x^e x^f x^g R^h{}_{bec} \nabla_h R^a{}_{fgd} - \frac{1}{9}\, x^e x^f x^g R^h{}_{bec} \nabla_f R^a{}_{hgd} + \frac{1}{36}\, x^e x^f x^g R^h{}_{bec} \nabla^a R_{hfgd} \Bigg) \\
& - \frac{1}{18}\, \Delta x^{\,b} \Delta x^{\,c} \Delta x^{\,d} \Delta x^{\,e} x^f x^g R^a{}_{hfb} \nabla_c R^h{}_{dge} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
x_3^a = \Bigg(& \Delta x^{\,b} \Delta x^{\,c} \Delta x^{\,d} \Bigg(-\frac{1}{12}\, x^e \nabla_b R^a{}_{ced} - \frac{1}{45}\, x^e x^f R^a{}_{egb} R_{gcfd} - \frac{4}{45}\, x^e x^f R^a{}_{bge} R_{gcfd} + \frac{1}{45}\, x^e x^f R^a{}_{bgc} R_{gefd} - \frac{1}{20}\, x^e x^f \nabla_{eb} R^a{}_{cfd} - \frac{1}{60}\, x^e x^f \nabla_{bc} R^a{}_{efd} \\
& + \frac{1}{15}\, x^e x^f R^a{}_{geb} R_{gcfd} - \frac{1}{120}\, x^e x^f \nabla^a{}_b R_{ecfd} - \frac{1}{54}\, x^e x^f x^g R_{hbec} \nabla_d R^a{}_{fhg} + \frac{1}{54}\, x^e x^f x^g R_{hbec} \nabla_h R^a{}_{fgd} - \frac{1}{27}\, x^e x^f x^g R_{hbec} \nabla_f R^a{}_{dhg} + \frac{1}{27}\, x^e x^f x^g R_{hbec} \nabla_f R^a{}_{hgd} \\
& - \frac{1}{54}\, x^e x^f x^g R_{hbec} \nabla^a R_{hfgd} - \frac{1}{54}\, x^e x^f x^g R^a{}_{bhe} \nabla_c R_{hfgd} - \frac{1}{27}\, x^e x^f x^g R^a{}_{bhe} \nabla_f R_{hcgd} - \frac{1}{108}\, x^e x^f x^g R^a{}_{bhe} \nabla_h R_{fcgd} + \frac{1}{54}\, x^e x^f x^g R^a{}_{heb} \nabla_c R_{hfgd} \\
& + \frac{1}{27}\, x^e x^f x^g R^a{}_{heb} \nabla_f R_{hcgd} + \frac{1}{108}\, x^e x^f x^g R^a{}_{heb} \nabla_h R_{fcgd} \Bigg) - \frac{1}{12}\, \Delta x^{\,b} \Delta x^{\,c} \Delta x^{\,d} \Delta x^{\,e} x^f x^g R^h{}_{bfc} \nabla_h R^a{}_{dge} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
x_4^a = \Delta x^{\,b} \Delta x^{\,c} \Delta x^{\,d} \Delta x^{\,e} \Bigg(& -\frac{1}{54}\, x^f R^a{}_{bgc} R_{gdfe} - \frac{1}{216}\, x^f x^g R^a{}_{bhc} \nabla_d R_{hfge} - \frac{1}{108}\, x^f x^g R^a{}_{bhc} \nabla_f R_{hdge} - \frac{1}{432}\, x^f x^g R^a{}_{bhc} \nabla_h R_{fdge} - \frac{1}{216}\, x^f x^g R_{hbfc} \nabla_d R^a{}_{ghe} \\
& + \frac{5}{216}\, x^f x^g R_{hbfc} \nabla_d R^a{}_{hge} - \frac{1}{108}\, x^f x^g R_{hbfc} \nabla_g R^a{}_{dhe} + \frac{1}{54}\, x^f x^g R_{hbfc} \nabla_h R^a{}_{dge} - \frac{1}{216}\, x^f x^g R_{hbfc} \nabla^a R_{hdge} - \frac{1}{36}\, x^f x^g R_{hbfc} \nabla_d R^a{}_{ehg} - \frac{1}{72}\, x^f x^g R^a{}_{bhf} \nabla_c R_{hdge} \\
& + \frac{1}{72}\, x^f x^g R^a{}_{hfb} \nabla_c R_{hdge} \Bigg)
\end{aligned}
$$

$$
x_5^a = \Delta x^{\,b} \Delta x^{\,c} \Delta x^{\,d} \Delta x^{\,e} \Delta x^{\,f} \left(-\frac{1}{180}\, x^g R^a{}_{bhc} \nabla_d R_{hegf} - \frac{1}{120}\, x^g R_{hbgc} \nabla_d R^a{}_{ehf} \right)
$$

Translated RNC frames

Let P and Q be two points with RNC coordinates x^a and $x^a + \Delta x^a$ relative to a third point O. At P we can construct a new set of RNC coordinates, y^a. We will rotate the y^a frame so that the x^a and y^a coordinate axes are aligned at P. Then the RNC coordinates of Q relative to P are

$$
\begin{aligned}
y^a = \Bigg(& \Delta x^{\,a} + \Delta x^{\,b} \Delta x^{\,c} \Bigg(\frac{1}{3}\, x^d R^a{}_{bdc} + \frac{1}{12}\, x^d x^e \nabla_b R^a{}_{dec} + \frac{1}{6}\, x^d x^e \nabla_d R^a{}_{bec} + \frac{1}{24}\, x^d x^e \nabla^a R_{dbec} + \frac{4}{45}\, x^d x^e x^f R^a{}_{dge} R_{gbfc} - \frac{2}{45}\, x^d x^e x^f R^a{}_{bgd} R_{gefc} - \frac{1}{45}\, x^d x^e x^f R^a{}_{dgb} R_{gefc} \\
& + \frac{1}{20}\, x^d x^e x^f \nabla_{db} R^a{}_{efc} + \frac{1}{20}\, x^d x^e x^f \nabla_{de} R^a{}_{bfc} + \frac{1}{45}\, x^d x^e x^f R^a{}_{gdb} R_{gefc} + \frac{1}{40}\, x^d x^e x^f \nabla^a{}_d R_{ebfc} + \frac{2}{45}\, x^d x^e x^f x^g R_{hbdc} \nabla_e R^a{}_{fhg} + \frac{1}{60}\, x^d x^e x^f x^g R^a{}_{dhe} \nabla_b R_{hfgc} \\
& + \frac{2}{45}\, x^d x^e x^f x^g R^a{}_{dhe} \nabla_f R_{hbgc} - \frac{1}{45}\, x^d x^e x^f x^g R^a{}_{bhd} \nabla_e R_{hfgc} - \frac{1}{90}\, x^d x^e x^f x^g R^a{}_{dhb} \nabla_e R_{hfgc} - \frac{1}{90}\, x^d x^e x^f x^g R_{hdeb} \nabla_c R^a{}_{fhg} - \frac{1}{45}\, x^d x^e x^f x^g R_{hdeb} \nabla_f R^a{}_{chg} \\
& - \frac{1}{90}\, x^d x^e x^f x^g R_{hdeb} \nabla_f R^a{}_{ghc} + \frac{1}{60}\, x^d x^e x^f x^g \nabla_{deb} R^a{}_{fgc} + \frac{1}{90}\, x^d x^e x^f x^g \nabla_{def} R^a{}_{bgc} + \frac{1}{72}\, x^d x^e x^f x^g R^a{}_{dhe} \nabla_h R_{fbgc} + \frac{1}{90}\, x^d x^e x^f x^g R^a{}_{hdb} \nabla_e R_{hfgc} \\
& - \frac{1}{90}\, x^d x^e x^f x^g R_{hdeb} \nabla^a R_{hfgc} + \frac{1}{90}\, x^d x^e x^f x^g R_{hdeb} \nabla_f R^a{}_{hgc} + \frac{1}{120}\, x^d x^e x^f x^g \nabla^a{}_{de} R_{fbgc} \Bigg) + \Delta x^{\,b} \Delta x^{\,c} \Delta x^{\,d} \Bigg(\frac{1}{12}\, x^e \nabla_b R^a{}_{ced} + \frac{1}{45}\, x^e x^f R^a{}_{egb} R_{gcfd} \\
& + \frac{4}{45}\, x^e x^f R^a{}_{bge} R_{gcfd} - \frac{1}{45}\, x^e x^f R^a{}_{bgc} R_{gefd} + \frac{1}{20}\, x^e x^f \nabla_{eb} R^a{}_{cfd} + \frac{1}{60}\, x^e x^f \nabla_{bc} R^a{}_{efd} + \frac{7}{45}\, x^e x^f R^a{}_{geb} R_{gcfd} + \frac{1}{120}\, x^e x^f \nabla^a{}_b R_{ecfd} + \frac{1}{54}\, x^e x^f x^g R_{hbec} \nabla_d R^a{}_{fhg} \\
& + \frac{1}{27}\, x^e x^f x^g R_{hbec} \nabla_h R^a{}_{fgd} + \frac{1}{27}\, x^e x^f x^g R_{hbec} \nabla_f R^a{}_{dhg} + \frac{2}{27}\, x^e x^f x^g R_{hbec} \nabla_f R^a{}_{hgd} - \frac{1}{108}\, x^e x^f x^g R_{hbec} \nabla^a R_{hfgd} + \frac{1}{54}\, x^e x^f x^g R^a{}_{bhe} \nabla_c R_{hfgd} \\
& + \frac{1}{27}\, x^e x^f x^g R^a{}_{bhe} \nabla_f R_{hcgd} + \frac{1}{108}\, x^e x^f x^g R^a{}_{bhe} \nabla_h R_{fcgd} + \frac{1}{27}\, x^e x^f x^g R^a{}_{heb} \nabla_c R_{hfgd} + \frac{2}{27}\, x^e x^f x^g R^a{}_{heb} \nabla_f R_{hcgd} + \frac{1}{54}\, x^e x^f x^g R^a{}_{heb} \nabla_h R_{fcgd} \Bigg) \\
& + \Delta x^{\,b} \Delta x^{\,c} \Delta x^{\,d} \Delta x^{\,e} \Bigg(\frac{1}{54}\, x^f R^a{}_{bgc} R_{gdfe} + \frac{1}{216}\, x^f x^g R^a{}_{bhc} \nabla_d R_{hfge} + \frac{1}{108}\, x^f x^g R^a{}_{bhc} \nabla_f R_{hdge} + \frac{1}{432}\, x^f x^g R^a{}_{bhc} \nabla_h R_{fdge} + \frac{1}{216}\, x^f x^g R_{hbfc} \nabla_d R^a{}_{ghe} \\
& - \frac{5}{216}\, x^f x^g R_{hbfc} \nabla_d R^a{}_{hge} + \frac{1}{108}\, x^f x^g R_{hbfc} \nabla_g R^a{}_{dhe} + \frac{7}{108}\, x^f x^g R_{hbfc} \nabla_h R^a{}_{dge} + \frac{1}{216}\, x^f x^g R_{hbfc} \nabla^a R_{hdge} + \frac{1}{36}\, x^f x^g R_{hbfc} \nabla_d R^a{}_{ehg} + \frac{1}{72}\, x^f x^g R^a{}_{bhf} \nabla_c R_{hdge} \\
& + \frac{1}{24}\, x^f x^g R^a{}_{hfb} \nabla_c R_{hdge} \Bigg) + \Delta x^{\,b} \Delta x^{\,c} \Delta x^{\,d} \Delta x^{\,e} \Delta x^{\,f} \left(\frac{1}{180}\, x^g R^a{}_{bhc} \nabla_d R_{hegf} + \frac{1}{120}\, x^g R_{hbgc} \nabla_d R^a{}_{ehf} \right) \Bigg) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

The geodesic length

The squared geodesic length between the points P and Q is given by

$$
\begin{aligned}
\left(\int_P^Q ds\right)^2 = \Big(& g_{ab}\Delta x^{a}\Delta x^{b} - \frac{1}{3} R_{abcd}x^a x^c \Delta x^{b}\Delta x^{d} - \frac{1}{12}\nabla_a R_{bcde}x^b x^d \Delta x^{a}\Delta x^{c}\Delta x^{e} - \frac{1}{6}\nabla_a R_{bcde}x^a x^b x^d \Delta x^{c}\Delta x^{e} + \frac{2}{45} R_{abcd}R_{aefg}x^b x^c x^f \Delta x^{d}\Delta x^{e}\Delta x^{g} \\
& - \frac{1}{20}\nabla_{ab}R_{cdef}x^a x^c x^e \Delta x^{b}\Delta x^{d}\Delta x^{f} - \frac{11}{45} R_{abcd}R_{aefg}x^c x^f \Delta x^{b}\Delta x^{d}\Delta x^{e}\Delta x^{g} - \frac{1}{60}\nabla_{ab}R_{cdef}x^c x^e \Delta x^{a}\Delta x^{b}\Delta x^{d}\Delta x^{f} + \frac{2}{45} R_{abcd}R_{aefg}x^b x^c x^e x^f \Delta x^{d}\Delta x^{g} \\
& - \frac{1}{20}\nabla_{ab}R_{cdef}x^a x^b x^c x^e \Delta x^{d}\Delta x^{f} + \frac{1}{45} R_{abcd}\nabla_e R_{afgh}x^c x^e x^f x^g \Delta x^{b}\Delta x^{d}\Delta x^{h} + \frac{1}{45} R_{abcd}\nabla_e R_{afgh}x^b x^c x^f x^g \Delta x^{d}\Delta x^{e}\Delta x^{h} \\
& + \frac{1}{45} R_{abcd}\nabla_e R_{afgh}x^b x^c x^e x^g \Delta x^{d}\Delta x^{f}\Delta x^{h} - \frac{1}{60}\nabla_{abc}R_{defg}x^a x^b x^d x^f \Delta x^{c}\Delta x^{e}\Delta x^{g} - \frac{2}{27} R_{abcd}\nabla_e R_{afgh}x^c x^f x^g \Delta x^{b}\Delta x^{d}\Delta x^{e}\Delta x^{h} \\
& - \frac{1}{12} R_{abcd}\nabla_a R_{efgh}x^c x^e x^g \Delta x^{b}\Delta x^{d}\Delta x^{f}\Delta x^{h} - \frac{5}{27} R_{abcd}\nabla_e R_{afgh}x^c x^e x^g \Delta x^{b}\Delta x^{d}\Delta x^{f}\Delta x^{h} + \frac{1}{54} R_{abcd}\nabla_e R_{afgh}x^c x^g \Delta x^{b}\Delta x^{d}\Delta x^{e}\Delta x^{f}\Delta x^{h} \\
& + \frac{1}{18} R_{abcd}\nabla_e R_{afgh}x^b x^c x^g \Delta x^{d}\Delta x^{e}\Delta x^{f}\Delta x^{h} + \frac{2}{45} R_{abcd}\nabla_e R_{afgh}x^b x^c x^e x^f x^g \Delta x^{d}\Delta x^{h} - \frac{1}{90}\nabla_{abc}R_{defg}x^a x^b x^c x^d x^f \Delta x^{e}\Delta x^{g} \Big) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$