Notation

I find it tiresome to continually read and write long tensor expressions such as

$$\Gamma^a{}_{bc,i_1 i_2 i_3 \cdots i_n} A^{i_1 i_2 i_3 \cdots i_n}$$

So I propose a small change in notation. I will use single greek letters to denote strings of indices while reserving roman letters for single indices. In this notation the previous line would be written as

$$\Gamma^a{}_{bc,\beta} A^\beta$$

The number of indices inside β is unknown but usually that would not be a problem. The number n in the old notation normally serves only to remind us that we have an unknown long series of indices. The value of n is usually unknown and of little concern. Thus it seems reasonable to remove n from the picture.

How might we deal with something like

$$\Gamma^a{}_{bc,i_1 i_2 i_3 \cdots i_n} A^{i_1} A^{i_2} A^{i_3} \cdots A^{i_n} \ ?$$

I propose writing

$$\Gamma^a{}_{bc,\beta} A^{\cdot\beta}$$

The single dot reminds us that we are to multiply n copies of the object A, one for each of the normal indices in β.

Here is another common construction

$$\left(\cdots \left(\left(\Gamma^a{}_{bc,i_1} A^{i_1} \right)_{,i_2} A^{i_2} \right)_{,i_3} A^{i_3} \cdots \right)_{,i_n} A^{i_n}$$

How might we tidy this up? By including a dot before the derivative index β, like this

$$\Gamma^a{}_{bc,.\beta} A^{\cdot\beta}$$

I had also thought of writing a bar underneath any letter to denote arbitrary length indices. The advantage is that it places no constraints on the choice of index and it provides a better visual clue than roman versus greek letters. Its main weakness is that it mght make the equation look a little clutterd. Maybe that's a weak objection. Here is how it would look

$$\Gamma^a{}_{bc,.\underline{d}} A^{\cdot\underline{d}}$$

Symmetrised covariant differentiation

Computing *symmetrised* covariant derivatives is a rather mundane process. The first two symmetrised covariant derivatives of a tensor such as v_a can be computed using

$$v_{a;b} = v_{a,b} - \Gamma^c_{ab} v_c$$

$$v_{a;(bc)} = v_{a,bc} - 2\Gamma^d_{a(b} v_{d,c)} - \Gamma^d_{bc} v_{a,d} + \Gamma^d_{a(b} \Gamma^e_{c)d} v_e + \Gamma^d_{ae} \Gamma^e_{bc} v_d - \Gamma^d_{a(c,b)} v_d$$

One can readily appreciate that the higher order computations are tedious and unwieldly. So the obvious question is : How might we train Cadabra to do these computations? One approach is simply to provide all the standard rules but there is (in my opinion) a very elegant trick that makes the

Cadabra code rather simple. The essence of the trick is to introduce carefully chosen fields so that the successive covariant derivatives can be calculated by doing little more than successive ordinary differentiation.

Consider a geodesic passing through a point P. Let s be the arc length measured from P and let $D^a(s)$ be the unit tangent vector on the geodesic. Now let $A^a(s)$ be a vector field on the geodesic such that its covariant derivative along the geodesic is zero, that is

$$0 = \nabla_D\, A^a = \frac{dA^a}{ds} + \Gamma^a{}_{bc} A^b D^c$$

Now contract the first equation above with $A^a D^b$,

$$\begin{aligned}
v_{a;b} A^a D^b &= v_{a,b} A^a D^b - \Gamma^c{}_{ab} v_c A^a D^b \\
&= v_{a,b} A^a D^b + v_a \frac{dA^a}{ds} \\
&= \frac{dv_a}{ds} A^a + v_a \frac{dA^a}{ds} \\
&= \frac{d\left(v_a A^a\right)}{ds}
\end{aligned}$$

Thus the first covariant derivative can be computed by one round of ordinary differentiation. This result also shows how we can compute higher order derivatives. Since $0 = \nabla_D D^a$ we can easily see that

$$v_{a;\underline{b}} A^a D^{\underline{b}} = \frac{d^n\left(v_a A^a\right)}{ds^n}$$

Thus any higher order covariant derivative can be obtained simply by expanding the right hind side, one derivative at a time, while using the parallel transport condition to eliminate derivatives in A^a and D^a. The Cadabra code for this is is much the same. Each successive covariant derivative is obtained by applying d/ds to the previous result then using substitutions to eliminate the newly introduced derivatives of A^a and D^a.

Riemann normal coordinates

Consider any point P in some space and some arbibrary geodesic through that point. If we choose Riemann normal coordinates, centred on P and covering a neighbourhood of P, then the gedoesic maybe written as

$$x^a(s) = sA^a$$

where A^a is a constant and s is the proper distance along the gedoesic from P to the typical point $x^a(s)$ on the geodesic. Evidently the coordinates have be chosen so that $x^a = 0$ identifies the point P.

Since $x^a(s)$ describes a geodesic, it must be a solution of the geodesic equation, that is

$$0 = \frac{d^2x^a}{ds^2} + \Gamma^a{}_{bc} \frac{dx^b}{ds}\frac{dx^c}{ds}$$

This applies everywhere on the geodesic. After repeated differentiations and using $0 = dA^a/ds$ we find

$$0 = \Gamma^a{}_{bc,\underline{a}} A^{\cdot\underline{a}}$$

This is not the usual form in which this result is found, usually it is written as

$$0 = \Gamma^a{}_{(bc,i_1 i_2 i_3 \cdots i_n)}$$

This last form is valid only at P whereas the previous equation applies everywhere along the geodesic.

You might wonder why this last result does not apply at all points along the geodesic. Consider any other point, say Q. Then there is only one geodesic joining P to Q. Thus at Q you do not have the freedom to argue that A^a is arbitrary and thus peel away the factors of A^a from the second last equation. The second last equation is true at Q but only for a specific choice of A^a.

Turn attention now to computing somthing like $d^n\left(v_{\underline{a}}A^{\cdot\underline{a}}\right)/ds^n$. Since we know that the derivatives d^nA^a/ds^n vanish at P in the RNC we have

$$\frac{d^n\left(v_{\underline{a}}A^{\cdot\underline{a}}\right)}{ds^n}=\frac{d^nv_{\underline{a}}}{ds^n}A^{\cdot\underline{a}}=v_{\underline{a};\cdot\underline{b}}A^{\cdot\underline{a}}A^{\cdot\underline{b}}=v_{\underline{a},\cdot\underline{b}}A^{\cdot\underline{a}}A^{\cdot\underline{b}}$$

where we have used $d(...)/ds=(...)_{;a}A^a=(...)_{,a}A^a$. Again this is valid along the geodesic but if we evaluate the expression at P we know that the A^a are arbitrary, thus we have

$$v_{(\underline{a};\underline{b})}=v_{(\underline{a},\underline{b})}$$

Symmetrised derivatives in Riemann normal coordinates

Here is a challenge : express the symmetrised higher order covariant derivatives of v_{ab} in terms of the partial derivatives of v_{ab} and the connection $\Gamma^a{}_{bc}$. The result becomes rather unwieldly after only a few derivatives are taken.

We will follow the method outlined above with the extra feature that we will adapt our computations to Riemann normal coordinates. Our starting point is the scalar function $Q=v_{ab}A^aA^b$. We will take higher ordre directional derivatives of Q along a geodesic through a point P. Let's be specific. Let A^a be the unit tangent to the geodesic and let s be the proper distance measured along the geodesic from P.

In Riemann normal coordinates the geodesic through P is described by

$$x^a(s)=sA^a$$

and the A^a are constant along the geodesic, that is, for all $n>0$

$$0=\frac{d^nA^a}{ds^n}$$

Also, since A^a is the unit tangent to the geodesic, we have

$$0=A^a{}_{;b}A^b$$

We will use the above extensively in the following calculations.

$$\begin{aligned}\frac{d^nQ}{ds^n}&=\frac{d^n}{ds^n}\left(v_{ab}A^aA^b\right)\\&=\frac{d^{n-1}}{ds^{n-1}}\left(v_{ab,c}A^aA^bA^c\right)\\&=v_{ab,\underline{c}}A^aA^bA^{\cdot\underline{c}}\end{aligned}$$

This result only applies in the Riemann normal frame at the point P.

We can now repeat the same calculation but this time using covariant derivatives rather than partial derivatives

$$\begin{aligned}\frac{d^nQ}{ds^n} &= \frac{d^n}{ds^n}\left(v_{ab}A^aA^b\right)\\ &= \frac{d^{n-1}}{ds^{n-1}}\left(v_{ab;c}A^aA^bA^c\right)\\ &= v_{ab;\underline{c}}A^aA^bA^{\cdot\underline{c}}\end{aligned}$$

This result is general, it applies in all frames at P.

Clearly both results must yield the same value at P, thus we must have

$$v_{ab;\underline{c}}A^aA^bA^{\cdot\underline{c}} = v_{ab,\underline{c}}A^aA^bA^{\cdot\underline{c}}$$

at P and only in the Riemann normal frame at P. Since the A^a may be freely chosen at P we see that

$$v_{(ab;\underline{c})} = v_{(ab,\underline{c})}$$

at P and in the Riemann normal frame.

Symmetrised derivatives of the Riemann tensor

Put $v_{ab} = R_{acbd}B^{cd}$ and choose B^{ab} to be parallel transported along the geodesic. Thus we have

$$\left(R_{apbq}B^{pq}\right)_{;\underline{c}}A^aA^bA^{\cdot\underline{c}} = \left(R_{apbq}B^{pq}\right)_{,\underline{c}}A^aA^bA^{\cdot\underline{c}}$$

But since $0 = B^{ab}{}_{;c}A^c$ we can drag the B^{pq} through the derivatives on the left hand side to obtain

$$R_{apbq;\underline{c}}B^{pq}A^aA^bA^{\cdot\underline{c}} = \left(R_{apbq}B^{pq}\right)_{,\underline{c}}A^aA^bA^{\cdot\underline{c}}$$

The right hand side can be expanded and the derivatives of B eliminated using the parallel transport condition. In this way this particular form of symmetrised covariant derivative of the Riemann tensor can be expressed soley in terms of its partial deriavtives and those of the connection.

When we expand the right hand side we will interpret the derivatives in two different ways. Every derivative is of the form $Q_{,c}A^c$. When acting on objects like R_{abcd} and $\Gamma^a{}_{bc}$ we use the derivative as written, that is, as a chain rule. But on objects like A^a and B^{ab} we will compute this derivative as a directional derivative, as follows

$$\begin{aligned}A^a{}_{,b}A^b &= \frac{dA^a}{ds} = 0\\ B^{ab}{}_{,c}A^c &= \frac{dB^{ab}}{ds} = -\Gamma^a{}_{dc}B^{db}A^c - \Gamma^b{}_{dc}B^{ad}A^c\end{aligned}$$

Can we get the inverse result? That is, can we express the symmetrised partial derivatives of the Riemann tensor in terms of its covariant derivatives? Yes, but this time we take B^{ab} to be constants along the geodesic. That is we have

$$\begin{aligned}A^a{}_{,b}A^b &= \frac{dA^a}{ds} = 0\\ B^{ab}{}_{,c}A^c &= \frac{dB^{ab}}{ds} = 0\\ B^{ab}{}_{;c}A^c &= \frac{dB^{ab}}{ds} + \Gamma^a{}_{dc}B^{db}A^c + \Gamma^b{}_{dc}B^{ad}A^c\end{aligned}$$

We can also retrace out steps, rerurning to

$$\left(R_{apbq}B^{pq}\right)_{;\underline{c}} A^a A^b A^{\cdot\underline{c}} = \left(R_{apbq}B^{pq}\right)_{,\underline{c}} A^a A^b A^{\cdot\underline{c}}$$

and since $dB^{ab}/ds = 0$ we see that

$$R_{apbq,\underline{c}}B^{pq} A^a A^b A^{\cdot\underline{c}} = \left(R_{apbq}B^{pq}\right)_{;\underline{c}} A^a A^b A^{\cdot\underline{c}}$$

(note that the equation has been swapped left to right). And once again we can expand the right hand side using the equations given above for the covariant derivatives of B^{ab}.