Covariant differentiation

Suppose we wish to compute the covariant derivative $v_{a;b}$ of v_a. Here is a well known and useful (for us) variation on an otherwise standard computation.

First construct the scalar $v_a A^a$ and then compute its derivative along some geodesic curve with tangent vector D^b and parametrised by arc length s. So we have

$$\frac{d\left(v_a A^a\right)}{ds} = \left(v_a A^a\right)_{;b} D^b = \left(v_a A^a\right)_{,b} D^b$$

We are free to make any choice we like for the field A^a so let us demand that A^a is parallel transported along the curve, $0 = A^a{}_{;b} D^b$. We can use this to expand the above equation, leading to

$$\begin{aligned}
v_{a;b} A^a D^b &= \left(v_a A^a\right)_{,b} D^b \\
&= v_{a,b} A^a D^b + v_a A^a{}_{,b} D^b \\
&= v_{a,b} A^a D^b - \Gamma^c{}_{ab} v_c A^a D^b
\end{aligned}$$

Now since A and D are arbitrary we can easily read off the familar expression

$$v_{a;b} = v_{a,b} - \Gamma^c_{ab} v_c$$

What is the point of all this? Simply, it gives us a way to use Cadabra to compute higher order covariant derivatives (by n-rounds of d/ds on $v_a A^a$). It also allows us to compute the symmetrised higher order covariant derivatives of any object, such as $v_{(\underline{a};\underline{b})}$ where $\underline{a}$ and $\underline{b}$ are any set of indices (by setting $A^a = D^a$).

Our first task is to train Cadabra to do these computations (well what else would we be doing?)

```
# --- some basic declarations ---------------------------------------------------------------------

::PostDefaultRules( @@collect_terms!(%) ).

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t}::Indices(position=fixed).

\partial{#}::PartialDerivative.
```

```
A^{a}::Depends(\partial).
B^{a}::Depends(\partial).
D^{a}::Depends(\partial).
v_{a b}::Depends(\partial).
\Gamma^{a}_{b c}::Depends(\partial).
\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

# --- compute the first 3 covariant derivatives  --------------------------------------------------

eq00:=v_{a b} A^{a} B^{b}:

eq01:=D^{c}\partial_{c}{@(eq00)}:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(D^{a}\partial_{a}{A^{b}} -> -\Gamma^{b}_{a c}A^{a}D^{c}):
@substitute!(%)(D^{a}\partial_{a}{B^{b}} -> -\Gamma^{b}_{a c}B^{a}D^{c}):
@substitute!(%)(D^{a}\partial_{a}{D^{b}} -> -\Gamma^{b}_{a c}D^{a}D^{c}):
@prodsort!(%):
@rename_dummies!(%):
@canonicalise!(%):

eq02:=D^{c}\partial_{c}{@(eq01)}:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(D^{a}\partial_{a}{A^{b}} -> -\Gamma^{b}_{a c}A^{a}D^{c}):
@substitute!(%)(D^{a}\partial_{a}{B^{b}} -> -\Gamma^{b}_{a c}B^{a}D^{c}):
@substitute!(%)(D^{a}\partial_{a}{D^{b}} -> -\Gamma^{b}_{a c}D^{a}D^{c}):
@prodsort!(%):
@rename_dummies!(%):
@canonicalise!(%):
```

```
eq03:=D^{c}\partial_{c}{@(eq02)}:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(D^{a}\partial_{a}{A^{b}} -> -\Gamma^{b}_{a c}A^{a}D^{c}):
@substitute!(%)(D^{a}\partial_{a}{B^{b}} -> -\Gamma^{b}_{a c}B^{a}D^{c}):
@substitute!(%)(D^{a}\partial_{a}{D^{b}} -> -\Gamma^{b}_{a c}D^{a}D^{c}):
@prodsort!(%):
@rename_dummies!(%):
@canonicalise!(%):

tmp00:=v_{a} A^{a}:

tmp01:=D^{c}\partial_{c}{@(tmp00)}:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(D^{a}\partial_{a}{A^{b}} -> -\Gamma^{b}_{a c}A^{a}D^{c}):
@substitute!(%)(D^{a}\partial_{a}{D^{b}} -> -\Gamma^{b}_{a c}D^{a}D^{c}):
@prodsort!(%):
@rename_dummies!(%):
@canonicalise!(%):

tmp02:=D^{c}\partial_{c}{@(tmp01)}:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(D^{a}\partial_{a}{A^{b}} -> -\Gamma^{b}_{a c}A^{a}D^{c}):
@substitute!(%)(D^{a}\partial_{a}{D^{b}} -> -\Gamma^{b}_{a c}D^{a}D^{c}):
@prodsort!(%):
@rename_dummies!(%):
@canonicalise!(%):

# --- tidy up and display the results  -------------------------------------------------------------
```

```
@factor_out!!(eq00){A^{a}}:
@factor_out!!(eq00){B^{a}}:
@factor_out!!(eq00){D^{a}};

@factor_out!!(eq01){A^{a}}:
@factor_out!!(eq01){B^{a}}:
@factor_out!!(eq01){D^{a}};

@factor_out!!(eq02){A^{a}}:
@factor_out!!(eq02){B^{a}}:
@factor_out!!(eq02){D^{a}};

@factor_out!!(eq03){A^{a}}:
@factor_out!!(eq03){B^{a}}:
@factor_out!!(eq03){D^{a}};

@factor_out!!(tmp00){A^{a}}:
@factor_out!!(tmp00){D^{a}};

@factor_out!!(tmp01){A^{a}}:
@factor_out!!(tmp01){D^{a}};

@factor_out!!(tmp02){A^{a}}:
@factor_out!!(tmp02){D^{a}};

@depprint!(tmp01);
@depprint!(tmp02);
```

$$eq00 \ := v_{ab} A^{a} B^{b}$$

$$eq01 \ := A^{a} B^{b} D^{c} (\partial_{c} v_{ab} - \Gamma^{d}{}_{ac} v_{db} - \Gamma^{d}{}_{bc} v_{ad})$$

$$eq02 := A^a B^b D^c D^d (-2\,\Gamma^e{}_{ac}\partial_d v_{eb} - 2\,\Gamma^e{}_{bc}\partial_d v_{ae} - \Gamma^e{}_{cd}\partial_e v_{ab} + \partial_{cd} v_{ab} + \Gamma^e{}_{ac}\Gamma^f{}_{de} v_{fb} + 2\,\Gamma^e{}_{ac}\Gamma^f{}_{bd} v_{ef} + \Gamma^e{}_{af}\Gamma^f{}_{cd} v_{eb} - \partial_c\Gamma^e{}_{ad} v_{eb} + \Gamma^e{}_{bc}\Gamma^f{}_{de} v_{af} + \Gamma^e{}_{bf}\Gamma^f{}_{cd} v_{ae} - \partial_c\Gamma^e{}_{bd} v_{ae})$$

$$eq03 := A^a B^b D^c D^d D^e (3\,\Gamma^f{}_{ac}\Gamma^g{}_{df}\partial_e v_{gb} + 6\,\Gamma^f{}_{ac}\Gamma^g{}_{bd}\partial_e v_{fg} + 3\,\Gamma^f{}_{ag}\Gamma^g{}_{cd}\partial_e v_{fb} + 3\,\Gamma^f{}_{ac}\Gamma^g{}_{de}\partial_g v_{fb} - 3\,\partial_c\Gamma^f{}_{ad}\partial_e v_{fb} - 3\,\Gamma^f{}_{ac}\partial_{de} v_{fb} + 3\,\Gamma^f{}_{bc}\Gamma^g{}_{df}\partial_e v_{ag} + 3\,\Gamma^f{}_{bg}\Gamma^g{}_{cd}\partial_e v_{af} + 3\,\Gamma^f{}_{bc}\Gamma^g{}_{de}\partial_g v_{af} - 3\,\partial_c\Gamma^f{}_{bd}\partial_e v_{af} - 3\,\Gamma^f{}_{bc}\partial_{de} v_{af} + 2\,\Gamma^f{}_{cd}\Gamma^g{}_{ef}\partial_g v_{ab} - \partial_c\Gamma^f{}_{de}\partial_f v_{ab} - 3\,\Gamma^f{}_{cd}\partial_{ef} v_{ab} + \partial_{cde} v_{ab} - \Gamma^f{}_{ac}\Gamma^g{}_{df}\Gamma^h{}_{eg} v_{hb} - 3\,\Gamma^f{}_{ac}\Gamma^g{}_{bd}\Gamma^h{}_{ef} v_{hg} - \Gamma^f{}_{ag}\Gamma^g{}_{cd}\Gamma^h{}_{ef} v_{hb} - 2\,\Gamma^f{}_{ac}\Gamma^g{}_{de}\Gamma^h{}_{fg} v_{hb} + \Gamma^f{}_{cg}\partial_d\Gamma^g{}_{ae} v_{fb} + 2\,\Gamma^f{}_{ac}\partial_d\Gamma^g{}_{ef} v_{gb} - 3\,\Gamma^f{}_{ac}\Gamma^g{}_{bd}\Gamma^h{}_{eg} v_{fh} - 3\,\Gamma^f{}_{ag}\Gamma^g{}_{cd}\Gamma^h{}_{be} v_{fh} - 3\,\Gamma^f{}_{ac}\Gamma^g{}_{bh}\Gamma^h{}_{de} v_{fg} + 3\,\Gamma^f{}_{bc}\partial_d\Gamma^g{}_{ae} v_{gf} + 3\,\Gamma^f{}_{ac}\partial_d\Gamma^g{}_{be} v_{fg} - 2\,\Gamma^f{}_{ag}\Gamma^g{}_{ch}\Gamma^h{}_{de} v_{fb} + 2\,\Gamma^f{}_{cd}\partial_e\Gamma^g{}_{af} v_{gb} + \Gamma^f{}_{ag}\partial_c\Gamma^g{}_{de} v_{fb} + \Gamma^f{}_{cd}\partial_f\Gamma^g{}_{ae} v_{gb} - \partial_{cd}\Gamma^f{}_{ae} v_{fb} - \Gamma^f{}_{bc}\Gamma^g{}_{df}\Gamma^h{}_{eg} v_{ah} - \Gamma^f{}_{bg}\Gamma^g{}_{cd}\Gamma^h{}_{ef} v_{ah} - 2\,\Gamma^f{}_{bc}\Gamma^g{}_{de}\Gamma^h{}_{fg} v_{ah} + \Gamma^f{}_{cg}\partial_d\Gamma^g{}_{be} v_{af} + 2\,\Gamma^f{}_{bc}\partial_d\Gamma^g{}_{ef} v_{ag} - 2\,\Gamma^f{}_{bg}\Gamma^g{}_{ch}\Gamma^h{}_{de} v_{af} + 2\,\Gamma^f{}_{cd}\partial_e\Gamma^g{}_{bf} v_{ag} + \Gamma^f{}_{bg}\partial_c\Gamma^g{}_{de} v_{af} + \Gamma^f{}_{cd}\partial_f\Gamma^g{}_{be} v_{ag} - \partial_{cd}\Gamma^f{}_{be} v_{af})$$

$$tmp00 := v_a A^a$$

$$tmp01 := A^a D^b (\partial_b v_a - \Gamma^c{}_{ab} v_c)$$

$$tmp02 := A^a D^b D^c (-2\,\Gamma^d{}_{ab}\partial_c v_d - \Gamma^d{}_{bc}\partial_d v_a + \partial_{bc} v_a + \Gamma^d{}_{ab}\Gamma^e{}_{cd} v_e + \Gamma^d{}_{ae}\Gamma^e{}_{bc} v_d - \partial_b\Gamma^d{}_{ac} v_d)$$