

## The inverse metric tensor in Riemann normal coordinates

Here we calculate the Riemann normal expansion of the inverse metric,  $g^{ab}$ , by developing the recursive sequences

$$g^{ab}{}_{,d\underline{e}} = -\left(g^{cb}\Gamma^a{}_{c(d),\underline{e}}\right) - \left(g^{ac}\Gamma^b{}_{c(d),\underline{e}}\right)$$

$$(n+3)\Gamma^a{}_{d(b,c\underline{e})} = (n+1)\left(R^a{}_{(bcd,\underline{e})} - \left(\Gamma^a{}_{(cf}\Gamma^f{}_{bd),\underline{e}}\right)\right)$$

for  $n = 1, 2, 3, \dots$ . Note in these equations that the (extended) index  $\underline{e}$  contains  $n$  normal indices.

We can then construct a Taylor series for the metric using

$$\begin{aligned} g^{ab}(x) &= g^{ab} + g^{ab}{}_{,c}x^c + \frac{1}{2!}g^{ab}{}_{,cd}x^cx^d + \frac{1}{3!}g^{ab}{}_{,cde}x^cx^dx^e + \dots \\ &= g^{ab} + \sum_{n=1}^{\infty} \frac{1}{n!} g^{ab}{}_{,\underline{c}} x^{\underline{c}} \end{aligned}$$

```
# =====

::KeepHistory(false).
::PostDefaultRules( @@collect_terms!(%), @@sumflatten!(%) ).

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v}::Indices.

\nabla_{\#}::PartialDerivative.
\partial_{\#}::PartialDerivative.

g^{a b}::Metric.
\delta^{a}_{b}::KroneckerDelta.

R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.
R^{a}_{b c}{}^d::RiemannTensor.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

g^{a b}::Depends(\partial).
```

```

R_{a b c d}::Depends(\partial).
R^{a}_{b c d}::Depends(\partial).
\Gamma^{a}_{b c}::Depends(\partial).

# =====
#   compute the derivatives
# =====

# --- derivatives of g^{ab} -----

gab00:=g^{a b}:

gab01:= - g^{c b}\Gamma^{a}_{c d} - g^{a c}\Gamma^{b}_{c d}:

gab02:=\partial_e{ @(gab01) }:
@distributed! (%):
@prodrule! (%):
@substitute! (%)(\partial_d{g^{a b}} -> @(gab01)):
@distributed! (%):

gab03:=\partial_f{ @(gab02) }:
@distributed! (%):
@prodrule! (%):
@substitute! (%)(\partial_d{g^{a b}} -> @(gab01)):
@distributed! (%):

gab04:=\partial_g{ @(gab03) }:
@distributed! (%):
@prodrule! (%):
@substitute! (%)(\partial_d{g^{a b}} -> @(gab01)):
@distributed! (%):

gab05:=\partial_h{ @(gab04) }:
@distributed! (%):
@prodrule! (%):
@substitute! (%)(\partial_d{g^{a b}} -> @(gab01)):
@distributed! (%):

# --- derivatives of \Gamma -----

```

```

eq01:= (1/3) A^{b}A^{c} ( R^{a}_{b c d} - \Gamma^{a}_{c e}\Gamma^{e}_{b d} ):

eq02:= (6/4) A^{a}\partial_{a}{ @(eq01) }:
@distributed! (%):
@prodrule! (%):
@unwrap! (%):
@distributed! (%):

eq03:= (12/10) A^{a}\partial_{a}{ @(eq02) }:
@distributed! (%):
@prodrule! (%):
@unwrap! (%):
@distributed! (%):

eq04:= (20/18) A^{a}\partial_{a}{ @(eq03) }:
@distributed! (%):
@prodrule! (%):
@unwrap! (%):
@distributed! (%):

eq05:= (30/28) A^{a}\partial_{a}{ @(eq04) }:
@distributed! (%):
@prodrule! (%):
@unwrap! (%):
@distributed! (%):

# =====
#   switch to Riemann normal coordinates
# =====

# --- temporarily replace all derivatives of \Gamma with derivatives of T -----
# --- derivatives of g^{ab}

@substitute! (gab02) (\partial_{d}{\Gamma^{a}_{b c}} -> T_{d}^{a}_{b c}):
@substitute! (gab03) (\partial_{d}{\Gamma^{a}_{b c}} -> T_{d}^{a}_{b c}):
@substitute! (gab04) (\partial_{d}{\Gamma^{a}_{b c}} -> T_{d}^{a}_{b c}):
@substitute! (gab05) (\partial_{d}{\Gamma^{a}_{b c}} -> T_{d}^{a}_{b c}):

@substitute! (gab03) (\partial_{d e}{\Gamma^{a}_{b c}} -> T_{d e}^{a}_{b c}):

```

```

@substitute!(gab04)(\partial_{d e}\{\Gamma^a_{b c}\} -> T_{d e}^a_{b c}):
@substitute!(gab05)(\partial_{d e}\{\Gamma^a_{b c}\} -> T_{d e}^a_{b c}):

@substitute!(gab04)(\partial_{d e f}\{\Gamma^a_{b c}\} -> T_{d e f}^a_{b c}):
@substitute!(gab05)(\partial_{d e f}\{\Gamma^a_{b c}\} -> T_{d e f}^a_{b c}):

@substitute!(gab05)(\partial_{d e f g}\{\Gamma^a_{b c}\} -> T_{d e f g}^a_{b c}):

# --- derivatives of \Gamma^a_{b c}

@substitute!(eq02)(\partial_d\{\Gamma^a_{b c}\} -> T_d^a_{b c}):
@substitute!(eq03)(\partial_d\{\Gamma^a_{b c}\} -> T_d^a_{b c}):
@substitute!(eq04)(\partial_d\{\Gamma^a_{b c}\} -> T_d^a_{b c}):
@substitute!(eq05)(\partial_d\{\Gamma^a_{b c}\} -> T_d^a_{b c}):

@substitute!(eq03)(\partial_{d e}\{\Gamma^a_{b c}\} -> T_{d e}^a_{b c}):
@substitute!(eq04)(\partial_{d e}\{\Gamma^a_{b c}\} -> T_{d e}^a_{b c}):
@substitute!(eq05)(\partial_{d e}\{\Gamma^a_{b c}\} -> T_{d e}^a_{b c}):

@substitute!(eq04)(\partial_{d e f}\{\Gamma^a_{b c}\} -> T_{d e f}^a_{b c}):
@substitute!(eq05)(\partial_{d e f}\{\Gamma^a_{b c}\} -> T_{d e f}^a_{b c}):

@substitute!(eq05)(\partial_{d e f g}\{\Gamma^a_{b c}\} -> T_{d e f g}^a_{b c}):

# --- now set \Gamma to zero -----

@substitute!!(gab01)(\Gamma^a_{b c} -> 0):
@substitute!!(gab02)(\Gamma^a_{b c} -> 0):
@substitute!!(gab03)(\Gamma^a_{b c} -> 0):
@substitute!!(gab04)(\Gamma^a_{b c} -> 0):
@substitute!!(gab05)(\Gamma^a_{b c} -> 0):

@substitute!!(eq01)(\Gamma^a_{b c} -> 0):
@substitute!!(eq02)(\Gamma^a_{b c} -> 0):
@substitute!!(eq03)(\Gamma^a_{b c} -> 0):
@substitute!!(eq04)(\Gamma^a_{b c} -> 0):
@substitute!!(eq05)(\Gamma^a_{b c} -> 0):

# --- now re-introduce the derivatives of \Gamma -----
# --- derivatives of g^{ab}

```

```

@substitute!(gab02)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):
@substitute!(gab03)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):
@substitute!(gab04)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):
@substitute!(gab05)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):

@substitute!(gab03)(T_{d e}^{a}_{b c} -> \partial_{d e}\{\Gamma^{a}_{b c}\}):
@substitute!(gab04)(T_{d e}^{a}_{b c} -> \partial_{d e}\{\Gamma^{a}_{b c}\}):
@substitute!(gab05)(T_{d e}^{a}_{b c} -> \partial_{d e}\{\Gamma^{a}_{b c}\}):

@substitute!(gab04)(T_{d e f}^{a}_{b c} -> \partial_{d e f}\{\Gamma^{a}_{b c}\}):
@substitute!(gab05)(T_{d e f}^{a}_{b c} -> \partial_{d e f}\{\Gamma^{a}_{b c}\}):

@substitute!(gab05)(T_{d e f g}^{a}_{b c} -> \partial_{d e f g}\{\Gamma^{a}_{b c}\}):

# --- derivatives of \Gamma^{a}_{b c}

@substitute!(eq02)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):
@substitute!(eq03)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):
@substitute!(eq04)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):
@substitute!(eq05)(T_{d}^{a}_{b c} -> \partial_{d}\{\Gamma^{a}_{b c}\}):

@substitute!(eq03)(T_{d e}^{a}_{b c} -> \partial_{d e}\{\Gamma^{a}_{b c}\}):
@substitute!(eq04)(T_{d e}^{a}_{b c} -> \partial_{d e}\{\Gamma^{a}_{b c}\}):
@substitute!(eq05)(T_{d e}^{a}_{b c} -> \partial_{d e}\{\Gamma^{a}_{b c}\}):

@substitute!(eq04)(T_{d e f}^{a}_{b c} -> \partial_{d e f}\{\Gamma^{a}_{b c}\}):
@substitute!(eq05)(T_{d e f}^{a}_{b c} -> \partial_{d e f}\{\Gamma^{a}_{b c}\}):

@substitute!(eq05)(T_{d e f g}^{a}_{b c} -> \partial_{d e f g}\{\Gamma^{a}_{b c}\}):

# =====
#   prepare first six terms in the Taylor series expansion of g^{ab}(x)
# =====

term0:= @(gab00): @distribute!(%):
term1:= @(gab01) A^d: @distribute!(%):
term2:= @(gab02) A^d A^e: @distribute!(%):
term3:= @(gab03) A^d A^e A^f: @distribute!(%):
term4:= @(gab04) A^d A^e A^f A^g: @distribute!(%):
term5:= @(gab05) A^d A^e A^f A^g A^h: @distribute!(%):

```

```

# --- imported from symm-riemann.lib -----
qderiv01:="import symm-riemann.lib pderiv01":
@run(qderiv01){"/Users/leo/local/sh/cdbfile"}:

qderiv02:="import symm-riemann.lib pderiv02":
@run(qderiv02){"/Users/leo/local/sh/cdbfile"}:

qderiv03:="import symm-riemann.lib pderiv03":
@run(qderiv03){"/Users/leo/local/sh/cdbfile"}:

qderiv04:="import symm-riemann.lib pderiv04":
@run(qderiv04){"/Users/leo/local/sh/cdbfile"}:

qderiv05:="import symm-riemann.lib pderiv05":
@run(qderiv05){"/Users/leo/local/sh/cdbfile"}:

# =====
#   replace partial derivs of \Gamma with products and derivs of Riemann tensor
# =====

@substitute!(eq03)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!(eq03)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!(eq04)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(eq02)):
@substitute!(eq04)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(eq02)):
@substitute!(eq04)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!(eq04)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!(eq05)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} -> @(eq03)):
@substitute!(eq05)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} -> @(eq03)):
@substitute!(eq05)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(eq02)):
@substitute!(eq05)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(eq02)):
@substitute!(eq05)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!(eq05)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

```

```

@substitute!(qderiv02)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!(qderiv02)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!(qderiv03)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(eq02)):
@substitute!(qderiv03)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(eq02)):
@substitute!(qderiv03)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!(qderiv03)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!!(qderiv04)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} -> @(eq03)):
@substitute!!(qderiv04)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} -> @(eq03)):
@substitute!!(qderiv04)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(eq02)):
@substitute!!(qderiv04)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(eq02)):
@substitute!!(qderiv04)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!!(qderiv04)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!(term2)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!(term2)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!(term3)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(eq02)):
@substitute!(term3)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(eq02)):
@substitute!(term3)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!(term3)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!!(term4)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} -> @(eq03)):
@substitute!!(term4)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} -> @(eq03)):
@substitute!!(term4)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(eq02)):
@substitute!!(term4)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(eq02)):
@substitute!!(term4)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!!(term4)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute! (%):

@substitute!!(term5)(A^{c}A^{b}A^{e}A^{f}A^{g}\partial_{c e f g}\{\Gamma^{a}_{d b}\} -> @(eq04)):
@substitute!!(term5)(A^{c}A^{b}A^{e}A^{f}A^{g}\partial_{c e f g}\{\Gamma^{a}_{b d}\} -> @(eq04)):

```

```

@substitute!!(term5)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{d b}\} -> @(eq03)):
@substitute!!(term5)(A^{c}A^{b}A^{e}A^{f}\partial_{c e f}\{\Gamma^{a}_{b d}\} -> @(eq03)):
@substitute!!(term5)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{d b}\} -> @(eq02)):
@substitute!!(term5)(A^{c}A^{b}A^{e}\partial_{c e}\{\Gamma^{a}_{b d}\} -> @(eq02)):
@substitute!!(term5)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{b d}\} -> @(eq01)):
@substitute!!(term5)(A^{c}A^{b}\partial_{c}\{\Gamma^{a}_{d b}\} -> @(eq01)):
@distribute!(%):

# =====
#   replace partial derivatives of Riemann tensor with its covariant derivatives
# =====

@substitute!(qderiv03)(A^{b}A^{d}A^{e}\partial_{e}\{R^{u}_{b d v}\} -> @(qderiv01)):
@distribute!(%):

@substitute!(qderiv04)(A^{b}A^{d}A^{e}A^{f}\partial_{e f}\{R^{u}_{b d v}\} -> @(qderiv02)):
@substitute!(qderiv04)(A^{b}A^{d}A^{e}\partial_{e}\{R^{u}_{b d v}\} -> @(qderiv01)):
@distribute!(%):

@eliminate_metric!(term1):
@prodsort!(%):
@rename_dummies!(%):
@canonicalise!(%):

@eliminate_metric!(term2):
@prodsort!(%):
@rename_dummies!(%):
@canonicalise!(%):

@substitute!(term3)(A^{b}A^{d}A^{e}\partial_{e}\{R^{u}_{b d v}\} -> @(qderiv01)):
@distribute!(%):
@eliminate_metric!(%):
@prodsort!(%):
@rename_dummies!(%):
@canonicalise!(%):

@substitute!(term4)(A^{b}A^{d}A^{e}A^{f}\partial_{e f}\{R^{u}_{b d v}\} -> @(qderiv02)):
@substitute!(term4)(A^{b}A^{d}A^{e}\partial_{e}\{R^{u}_{b d v}\} -> @(qderiv01)):
@distribute!(%):

```

```

@eliminate_metric! (%):
@prodsort! (%):
@rename_dummies! (%):
@canonicalise! (%):

@substitute! (term5) (A^{b}A^{d}A^{e}A^{f}A^{g}\partial_{e f g}\{R^{u}_{b d v}\} -> @(qderiv03)):
@substitute! (term5) (A^{b}A^{d}A^{e}A^{f}\partial_{e f}\{R^{u}_{b d v}\} -> @(qderiv02)):
@substitute! (term5) (A^{b}A^{d}A^{e}\partial_{e}\{R^{u}_{b d v}\} -> @(qderiv01)):
@distribute! (%):
@eliminate_metric! (%):
@prodsort! (%):
@rename_dummies! (%):
@canonicalise! (%):

# =====
#   force all indices on R to be downstairs
# =====

@substitute! (term0) (R^{a}_{b c d} -> R_{a b c d},
    R_{a}^{b}_{c d} -> R_{a b c d},
    R^{a}_{b c}^{d} -> R_{a b c d},
    R_{a b}^{c}_{d} -> R_{a b c d},
    R_{a b c}^{d} -> R_{a b c d},
    R^{a}_{b}^{c}_{d} -> R_{a b c d},
    \nabla^{a}\{R_{b c d e}\} -> \nabla_{a}\{R_{b c d e}\},
    \nabla^{a}_{f}\{R_{b c d e}\} -> \nabla_{a f}\{R_{b c d e}\},
    \nabla^{a}_{f g}\{R_{b c d e}\} -> \nabla_{a f g}\{R_{b c d e}\}):

@substitute! (term1) (R^{a}_{b c d} -> R_{a b c d},
    R_{a}^{b}_{c d} -> R_{a b c d},
    R^{a}_{b c}^{d} -> R_{a b c d},
    R_{a b}^{c}_{d} -> R_{a b c d},
    R_{a b c}^{d} -> R_{a b c d},
    R^{a}_{b}^{c}_{d} -> R_{a b c d},
    \nabla^{a}\{R_{b c d e}\} -> \nabla_{a}\{R_{b c d e}\},
    \nabla^{a}_{f}\{R_{b c d e}\} -> \nabla_{a f}\{R_{b c d e}\},
    \nabla^{a}_{f g}\{R_{b c d e}\} -> \nabla_{a f g}\{R_{b c d e}\}):

@substitute! (term2) (R^{a}_{b c d} -> R_{a b c d},

```

```

R_{a}^{b}_{c d} -> R_{a b c d},
R^{a}_{b c}_{d} -> R_{a b c d},
R_{a b}^{c}_{d} -> R_{a b c d},
R_{a b c}^{d} -> R_{a b c d},
R^{a}_{b}^{c}_{d} -> R_{a b c d},
\nabla^{a}_{R_{b c d e}} -> \nabla_{a}{R_{b c d e}},
\nabla^{a}_{f}_{R_{b c d e}} -> \nabla_{a f}{R_{b c d e}},
\nabla^{a}_{f g}_{R_{b c d e}} -> \nabla_{a f g}{R_{b c d e}}):

```

```

@substitute!(term3)(R^{a}_{b c d} -> R_{a b c d},
R_{a}^{b}_{c d} -> R_{a b c d},
R^{a}_{b c}_{d} -> R_{a b c d},
R_{a b}^{c}_{d} -> R_{a b c d},
R_{a b c}^{d} -> R_{a b c d},
R^{a}_{b}^{c}_{d} -> R_{a b c d},
\nabla^{a}_{R_{b c d e}} -> \nabla_{a}{R_{b c d e}},
\nabla^{a}_{f}_{R_{b c d e}} -> \nabla_{a f}{R_{b c d e}},
\nabla^{a}_{f g}_{R_{b c d e}} -> \nabla_{a f g}{R_{b c d e}}):

```

```

@substitute!(term4)(R^{a}_{b c d} -> R_{a b c d},
R_{a}^{b}_{c d} -> R_{a b c d},
R^{a}_{b c}_{d} -> R_{a b c d},
R_{a b}^{c}_{d} -> R_{a b c d},
R_{a b c}^{d} -> R_{a b c d},
R^{a}_{b}^{c}_{d} -> R_{a b c d},
\nabla^{a}_{R_{b c d e}} -> \nabla_{a}{R_{b c d e}},
\nabla^{a}_{f}_{R_{b c d e}} -> \nabla_{a f}{R_{b c d e}},
\nabla^{a}_{f g}_{R_{b c d e}} -> \nabla_{a f g}{R_{b c d e}}):

```

```

@substitute!(term5)(R^{a}_{b c d} -> R_{a b c d},
R_{a}^{b}_{c d} -> R_{a b c d},
R^{a}_{b c}_{d} -> R_{a b c d},
R_{a b}^{c}_{d} -> R_{a b c d},
R_{a b c}^{d} -> R_{a b c d},
R^{a}_{b}^{c}_{d} -> R_{a b c d},
\nabla^{a}_{R_{b c d e}} -> \nabla_{a}{R_{b c d e}},
\nabla^{a}_{f}_{R_{b c d e}} -> \nabla_{a f}{R_{b c d e}},
\nabla^{a}_{f g}_{R_{b c d e}} -> \nabla_{a f g}{R_{b c d e}}):

```

```

# =====
#   raise indices {ab}
# =====

tmp01:=@(term0) g^{a u} g^{b v}: @distribute!(%): @eliminate_metric!(%):
tmp02:=@(tmp01) \delta^{a}_{u} \delta^{b}_{v}: @distribute!(%): @eliminate_kr!(%):
term0:=@(tmp02):

tmp01:=@(term1) g^{a u} g^{b v}: @distribute!(%): @eliminate_metric!(%):
tmp02:=@(tmp01) \delta^{a}_{u} \delta^{b}_{v}: @distribute!(%): @eliminate_kr!(%):
term1:=@(tmp02):

tmp01:=@(term2) g^{a u} g^{b v}: @distribute!(%): @eliminate_metric!(%):
tmp02:=@(tmp01) \delta^{a}_{u} \delta^{b}_{v}: @distribute!(%): @eliminate_kr!(%):
term2:=@(tmp02):

tmp01:=@(term3) g^{a u} g^{b v}: @distribute!(%): @eliminate_metric!(%):
tmp02:=@(tmp01) \delta^{a}_{u} \delta^{b}_{v}: @distribute!(%): @eliminate_kr!(%):
term3:=@(tmp02):

tmp01:=@(term4) g^{a u} g^{b v}: @distribute!(%): @eliminate_metric!(%):
tmp02:=@(tmp01) \delta^{a}_{u} \delta^{b}_{v}: @distribute!(%): @eliminate_kr!(%):
term4:=@(tmp02):

tmp01:=@(term5) g^{a u} g^{b v}: @distribute!(%): @eliminate_metric!(%):
tmp02:=@(tmp01) \delta^{a}_{u} \delta^{b}_{v}: @distribute!(%): @eliminate_kr!(%):
term5:=@(tmp02):

```

```
# =====
#   the metric expressed in terms of the Riemann tensor and its covariant derivatives
# =====

metric:=@(term0)
+ (1/1) @(term1)
+ (1/2) @(term2)
+ (1/6) @(term3)
+ (1/24) @(term4)
+ (1/120) @(term5):

@substitute!(%)(A^{a} -> x^{a}):
@canonicalise!(%); "metric.del"

@print["\Btag{01}g^{\{ab\}}(x)="\~@(metric)~"+\Big0{\eps^6}\Etag{01}"];
```

$$g^{ab}(x) = \left( g^{ab} + \frac{1}{3} x^c x^d R^a{}_c{}^b{}_d + \frac{1}{6} x^c x^d x^e \nabla_e R^a{}_d{}^b{}_e + \frac{1}{15} x^c x^d x^e x^f R^a{}_{cdg} R^b{}_{efg} + \frac{1}{20} x^c x^d x^e x^f \nabla_{cd} R^a{}_e{}^b{}_f + \frac{1}{30} x^c x^d x^e x^f x^g R^a{}_{cdh} \nabla_e R^b{}_{fgh} \right. \\ \left. + \frac{1}{30} x^c x^d x^e x^f x^g R^b{}_{cdh} \nabla_e R^a{}_{fgh} + \frac{1}{90} x^c x^d x^e x^f x^g \nabla_{cde} R^a{}_f{}^b{}_g \right) + \mathcal{O}(\epsilon^6)$$

```
# =====
#   rewrite the metric in the form  $g^{ab}(x) = g^{ab} + g^{ab}{}_c x^c + g^{ab}{}_{cd} x^c x^d + g^{ab}{}_{cde} x^c x^d x^e + \dots$ 
# =====

gab:=          @(term0):
gabc:=         @(term1):
gabacd:=       1/2  @(term2):
gabacde:=      1/6  @(term3):
gabacdef:=     1/24  @(term4):
gabacdefg:=    1/120 @(term5):

@substitute!(gab)(A^{a} -> 1): @sym!({^a,^b}):
@substitute!(gabc)(A^{a} -> 1): @sym!({^a,^b}):
@substitute!(gabacd)(A^{a} -> 1): @sym!({^a,^b}):
@substitute!(gabacde)(A^{a} -> 1): @sym!({^a,^b}):
@substitute!(gabacdef)(A^{a} -> 1): @sym!({^a,^b}):
@substitute!(gabacdefg)(A^{a} -> 1): @sym!({^a,^b}):
```

```
@print["g^{ab}-{ }_{c}"~@(gabc)];  
@print["g^{ab}-{ }_{cd}"~@(gabcd)];  
@print["g^{ab}-{ }_{cde}"~@(gabcde)];  
@print["g^{ab}-{ }_{cdef}"~@(gabcdef)];  
@print["g^{ab}-{ }_{cdefg}"~@(gabcdefg)];
```

```
# =====
#   the upstairs metric
#   =====
```

$$g^{ab}(x) = g^{ab} + g^{ab}{}_c x^c + g^{ab}{}_{cd} x^c x^d + g^{ab}{}_{cde} x^c x^d x^e + g^{ab}{}_{cdef} x^c x^d x^e x^f + g^{ab}{}_{cdefg} x^c x^d x^e x^f x^g + \mathcal{O}(\epsilon^6)$$

$$g^{ab}{}_c = 0$$

$$g^{ab}{}_{cd} = \left( \frac{1}{6} R^a{}_c{}^b{}_d + \frac{1}{6} R^b{}_c{}^a{}_d \right)$$

$$g^{ab}{}_{cde} = \left( \frac{1}{12} \nabla_c R^a{}_d{}^b{}_e + \frac{1}{12} \nabla_c R^b{}_d{}^a{}_e \right)$$

$$g^{ab}{}_{cdef} = \left( \frac{1}{30} R^a{}_{cdg} R^b{}_{efg} + \frac{1}{30} R^b{}_{cdg} R^a{}_{efg} + \frac{1}{40} \nabla_{cd} R^a{}_{e}{}^b{}_f + \frac{1}{40} \nabla_{cd} R^b{}_{e}{}^a{}_f \right)$$

$$g^{ab}{}_{cdefg} = \left( \frac{1}{30} R^a{}_{cdh} \nabla_e R^b{}_{fgh} + \frac{1}{30} R^b{}_{cdh} \nabla_e R^a{}_{fgh} + \frac{1}{180} \nabla_{cde} R^a{}_{f}{}^b{}_g + \frac{1}{180} \nabla_{cde} R^b{}_{f}{}^a{}_g \right)$$

```
# =====  
#   export metric  
# =====  
  
com:="open metric-inv.lib":  
@run(com){"/Users/leo/local/sh/cdbfile"}:  
  
com:="export metric-inv.lib metric.del":  
@run(com){"/Users/leo/local/sh/cdbfile"}:
```