Here we compute the *partial* derivatives of $R^a{}_{bcd}$ in terms of its *covariant* derivatives *directly* – i.e. without first computing partial derivatives and then doing substitutions (as in symm-riem-slow.cdbp).

$$R^p{}_{abq,\underline{c}}B^q{}_p A^a A^b A^{\cdot\underline{c}} = (R^p{}_{abq}B^q{}_p)_{;\underline{c}} A^a A^b A^{\cdot\underline{c}}$$

and

$$A^a{}_{;b}A^b = \frac{DA^a}{Ds} = 0$$
$$B^a{}_{b,c}A^c = \frac{dB^a{}_b}{ds} = 0$$
$$B^a{}_{b;c}A^c = \Gamma^a{}_{dc}B^d{}_b A^c - \Gamma^d{}_{bc}B^a{}_d A^c$$

This approach is much much quicker than that used in symm-riem-slow.cdbp, the first four derivatives took about 2sec while five derivative took about 17 secs. Compare this with 19 secs and 10 mins respectively for symm-riem-slow.cdbp

```
::PostDefaultRules( @@collect_terms!(%), @@sumflatten!(%) ).

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w#}::Indices.

# we can get away with using PartialDerivative for \nabla because we are computing
# symmetrised derivatives, this makes Cadabra's job much easier

\nabla{#}::PartialDerivative.
\partial{#}::PartialDerivative.

\delta^{a}_{b}::KroneckerDelta.

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

# --- compute the symmetric covariant derivatives of R^{a}_{bcd} B^{d}_{a} ------------------------

pderiv00:=R^{a}_{b c d} B^{d}_{a} A^{b} A^{c}:

pderiv01:=A^{a}\nabla_{a}{ @(pderiv00) }:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(\nabla_{a}{A^{b}} -> 0):
```

```
@canonicalise!(%):

pderiv02:=A^{a}\nabla_{a}{ @(pderiv01) }:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(\nabla_{a}{A^{b}} -> 0):
@canonicalise!(%):

pderiv03:=A^{a}\nabla_{a}{ @(pderiv02) }:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(\nabla_{a}{A^{b}} -> 0):
@canonicalise!(%):

pderiv04:=A^{a}\nabla_{a}{ @(pderiv03) }:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(\nabla_{a}{A^{b}} -> 0):
@canonicalise!(%):

pderiv05:=A^{a}\nabla_{a}{ @(pderiv04) }:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(\nabla_{a}{A^{b}} -> 0):
@canonicalise!(%):

# --- compute the covariant derivatives of B^{a}_{b}, note B^{a}_{b,c} is zero, by choice ---------
# --- this method of computing partial derivatives does not use auxillary fields ------------------

bderiv00:=B^{a}_{b}:

bderiv01:=A^{c}\partial_{c}{ @(bderiv00) } + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
                                           - \Gamma^{p}_{b q} W^{a}_{p} A^{q}:

@distribute!(%):
@prodrule!(%):
```

```
@distribute!(%):
@substitute!(%)(\partial_{a}{A^{b}} -> 0):
@substitute!(%)(\partial_{a}{B^{b}_{c}} -> 0):
@substitute!(%)(W^{a}_{b} -> @(bderiv00)):
@distribute!(%):
@canonicalise!(%):

bderiv02:=A^{c}\partial_{c}{ @(bderiv01) } + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
                                           - \Gamma^{p}_{b q} W^{a}_{p} A^{q}:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(\partial_{a}{A^{b}} -> 0):
@substitute!(%)(\partial_{a}{B^{b}_{c}} -> 0):
@substitute!(%)(W^{a}_{b} -> @(bderiv01)):
@distribute!(%):
@canonicalise!(%):

bderiv03:=A^{c}\partial_{c}{ @(bderiv02) } + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
                                           - \Gamma^{p}_{b q} W^{a}_{p} A^{q}:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(\partial_{a}{A^{b}} -> 0):
@substitute!(%)(\partial_{a}{B^{b}_{c}} -> 0):
@substitute!(%)(W^{a}_{b} -> @(bderiv02)):
@distribute!(%):
@canonicalise!(%):

bderiv04:=A^{c}\partial_{c}{ @(bderiv03) } + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
                                           - \Gamma^{p}_{b q} W^{a}_{p} A^{q}:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(\partial_{a}{A^{b}} -> 0):
@substitute!(%)(\partial_{a}{B^{b}_{c}} -> 0):
@substitute!(%)(W^{a}_{b} -> @(bderiv03)):
@distribute!(%):
```

```
@canonicalise!(%):

bderiv05:=A^{c}\partial_{c}{ @(bderiv04) } + \Gamma^{a}_{p q} W^{p}_{b} A^{q}
                                          - \Gamma^{p}_{b q} W^{a}_{p} A^{q}:
@distribute!(%):
@prodrule!(%):
@distribute!(%):
@substitute!(%)(\partial_{a}{A^{b}} -> 0):
@substitute!(%)(\partial_{a}{B^{b}_{c}} -> 0):
@substitute!(%)(W^{a}_{b} -> @(bderiv04)):
@distribute!(%):
@canonicalise!(%):
```

--- now turn to Riemann normal coordinates, defined by $0 = \Gamma^{a}_{(bc,\underline{a})}$ --------------

```
# --- temporarily replace all derivatives of \Gamma with derivatives of T ------------------------

@substitute!(bderiv01)(\partial_{d}{\Gamma^{a}_{b c}} -> T_{d}^{a}_{b c}):
@substitute!(bderiv02)(\partial_{d}{\Gamma^{a}_{b c}} -> T_{d}^{a}_{b c}):
@substitute!(bderiv03)(\partial_{d}{\Gamma^{a}_{b c}} -> T_{d}^{a}_{b c}):
@substitute!(bderiv04)(\partial_{d}{\Gamma^{a}_{b c}} -> T_{d}^{a}_{b c}):
@substitute!(bderiv05)(\partial_{d}{\Gamma^{a}_{b c}} -> T_{d}^{a}_{b c}):

@substitute!(bderiv02)(\partial_{d e}{\Gamma^{a}_{b c}} -> T_{d e}^{a}_{b c}):
@substitute!(bderiv03)(\partial_{d e}{\Gamma^{a}_{b c}} -> T_{d e}^{a}_{b c}):
@substitute!(bderiv04)(\partial_{d e}{\Gamma^{a}_{b c}} -> T_{d e}^{a}_{b c}):
@substitute!(bderiv05)(\partial_{d e}{\Gamma^{a}_{b c}} -> T_{d e}^{a}_{b c}):

@substitute!(bderiv03)(\partial_{d e f}{\Gamma^{a}_{b c}} -> T_{d e f}^{a}_{b c}):
@substitute!(bderiv04)(\partial_{d e f}{\Gamma^{a}_{b c}} -> T_{d e f}^{a}_{b c}):
@substitute!(bderiv05)(\partial_{d e f}{\Gamma^{a}_{b c}} -> T_{d e f}^{a}_{b c}):

@substitute!(bderiv04)(\partial_{d e f g}{\Gamma^{a}_{b c}} -> T_{d e f g}^{a}_{b c}):
@substitute!(bderiv05)(\partial_{d e f g}{\Gamma^{a}_{b c}} -> T_{d e f g}^{a}_{b c}):

@substitute!(bderiv05)(\partial_{d e f g h}{\Gamma^{a}_{b c}} -> T_{d e f g h}^{a}_{b c}):

# --- now set \Gamma to zero -------------------------------------------------------------------

@substitute!!(bderiv00)(\Gamma^{a}_{b c} -> 0):
```

```
@substitute!!(bderiv01)(\Gamma^{a}_{b c} -> 0):
@substitute!!(bderiv02)(\Gamma^{a}_{b c} -> 0):
@substitute!!(bderiv03)(\Gamma^{a}_{b c} -> 0):
@substitute!!(bderiv04)(\Gamma^{a}_{b c} -> 0):
@substitute!!(bderiv05)(\Gamma^{a}_{b c} -> 0):

# --- now re-introduce the derivatives of \Gamma ---------------------------------------------------

@substitute!(bderiv01)(T_{d}^{a}_{b c} -> \partial_{d}{\Gamma^{a}_{b c}}):
@substitute!(bderiv02)(T_{d}^{a}_{b c} -> \partial_{d}{\Gamma^{a}_{b c}}):
@substitute!(bderiv03)(T_{d}^{a}_{b c} -> \partial_{d}{\Gamma^{a}_{b c}}):
@substitute!(bderiv04)(T_{d}^{a}_{b c} -> \partial_{d}{\Gamma^{a}_{b c}}):
@substitute!(bderiv05)(T_{d}^{a}_{b c} -> \partial_{d}{\Gamma^{a}_{b c}}):

@substitute!(bderiv02)(T_{d e}^{a}_{b c} -> \partial_{d e}{\Gamma^{a}_{b c}}):
@substitute!(bderiv03)(T_{d e}^{a}_{b c} -> \partial_{d e}{\Gamma^{a}_{b c}}):
@substitute!(bderiv04)(T_{d e}^{a}_{b c} -> \partial_{d e}{\Gamma^{a}_{b c}}):
@substitute!(bderiv05)(T_{d e}^{a}_{b c} -> \partial_{d e}{\Gamma^{a}_{b c}}):

@substitute!(bderiv03)(T_{d e f}^{a}_{b c} -> \partial_{d e f}{\Gamma^{a}_{b c}}):
@substitute!(bderiv04)(T_{d e f}^{a}_{b c} -> \partial_{d e f}{\Gamma^{a}_{b c}}):
@substitute!(bderiv05)(T_{d e f}^{a}_{b c} -> \partial_{d e f}{\Gamma^{a}_{b c}}):

@substitute!(bderiv04)(T_{d e f g}^{a}_{b c} -> \partial_{d e f g}{\Gamma^{a}_{b c}}):
@substitute!(bderiv05)(T_{d e f g}^{a}_{b c} -> \partial_{d e f g}{\Gamma^{a}_{b c}}):

@substitute!(bderiv05)(T_{d e f g h}^{a}_{b c} -> \partial_{d e f g h}{\Gamma^{a}_{b c}}):

# --- substitute cov. derivs of B^{a}_{b} into cov. derivs of R^{a}_{bcd}B^{d}_{a} ----------------

@substitute!(pderiv01)(A^{c}\nabla_{c}{B^{a}_{b}} -> @(bderiv01)): @distribute!(%):
@substitute!(pderiv02)(A^{c}\nabla_{c}{B^{a}_{b}} -> @(bderiv01)): @distribute!(%):
@substitute!(pderiv03)(A^{c}\nabla_{c}{B^{a}_{b}} -> @(bderiv01)): @distribute!(%):
@substitute!(pderiv04)(A^{c}\nabla_{c}{B^{a}_{b}} -> @(bderiv01)): @distribute!(%):
@substitute!(pderiv05)(A^{c}\nabla_{c}{B^{a}_{b}} -> @(bderiv01)): @distribute!(%):

@substitute!(pderiv02)(A^{c}A^{d}\nabla_{c d}{B^{a}_{b}} -> @(bderiv02)): @distribute!(%):
@substitute!(pderiv03)(A^{c}A^{d}\nabla_{c d}{B^{a}_{b}} -> @(bderiv02)): @distribute!(%):
@substitute!(pderiv04)(A^{c}A^{d}\nabla_{c d}{B^{a}_{b}} -> @(bderiv02)): @distribute!(%):
@substitute!(pderiv05)(A^{c}A^{d}\nabla_{c d}{B^{a}_{b}} -> @(bderiv02)): @distribute!(%):
```

```
@substitute!(pderiv03)(A^{c}A^{d}A^{e}\nabla_{c d e}{B^{a}_{b}} -> @(bderiv03)): @distribute!(%):
@substitute!(pderiv04)(A^{c}A^{d}A^{e}\nabla_{c d e}{B^{a}_{b}} -> @(bderiv03)): @distribute!(%):
@substitute!(pderiv05)(A^{c}A^{d}A^{e}\nabla_{c d e}{B^{a}_{b}} -> @(bderiv03)): @distribute!(%):

@substitute!(pderiv04)(A^{c}A^{d}A^{e}A^{f}\nabla_{c d e f}{B^{a}_{b}} -> @(bderiv04)): @distribute!(%):
@substitute!(pderiv05)(A^{c}A^{d}A^{e}A^{f}\nabla_{c d e f}{B^{a}_{b}} -> @(bderiv04)): @distribute!(%):

@substitute!(pderiv05)(A^{c}A^{d}A^{e}A^{f}A^{g}\nabla_{c d e f g}{B^{a}_{b}} -> @(bderiv05)): @distribute!(%):

@prodsort!(pderiv01): @rename_dummies!(%): @factor_out!!(%){A^{a}}:
@prodsort!(pderiv02): @rename_dummies!(%): @factor_out!!(%){A^{a}}:
@prodsort!(pderiv03): @rename_dummies!(%): @factor_out!!(%){A^{a}}:
@prodsort!(pderiv04): @rename_dummies!(%): @factor_out!!(%){A^{a}}:
@prodsort!(pderiv05): @rename_dummies!(%): @factor_out!!(%){A^{a}}:

# --- done with B, so get rid of it ---------------------------------------------------------------
# --- this trick works only when 1) each term has only one factor of B^{a}_{b}
#                                2) the indices u,v do not appear anywhere in the expression

@substitute!(pderiv01)(B^{a}_{b} -> \delta^{a}_{v}\delta^{u}_{b}): @eliminate_kr!(%):
@substitute!(pderiv02)(B^{a}_{b} -> \delta^{a}_{v}\delta^{u}_{b}): @eliminate_kr!(%):
@substitute!(pderiv03)(B^{a}_{b} -> \delta^{a}_{v}\delta^{u}_{b}): @eliminate_kr!(%):
@substitute!(pderiv04)(B^{a}_{b} -> \delta^{a}_{v}\delta^{u}_{b}): @eliminate_kr!(%):
@substitute!(pderiv05)(B^{a}_{b} -> \delta^{a}_{v}\delta^{u}_{b}): @eliminate_kr!(%):

# --- tidy up and display the results  ------------------------------------------------------------

@print["\pderivA="~@(pderiv01)];
@print["\pderivB="~@(pderiv02)];
@print["\pderivC="~@(pderiv03)];
@print["\pderivD="~@(pderiv04)];
@print["\pderivE="~@(pderiv05)];
```

The first four symmetrised *covariant* derivatives of R^a_{bcd} in a Riemann normal coordinate frame.

$$A^a A^b A^c R^u{}_{bcv,a} = A^a A^b A^c \nabla_a R^u{}_{bcv}$$

$$A^a A^b A^c A^d R^u{}_{cdv,ab} = A^a A^b A^c A^d \left(\nabla_{ab} R^u{}_{cdv} + R^u{}_{abg} \partial_c \Gamma^g{}_{vd} - R^g{}_{abv} \partial_c \Gamma^u{}_{gd}\right)$$

$$A^a A^b A^c A^d A^e R^u{}_{dev,abc} = A^a A^b A^c A^d A^e \left(\nabla_{abc} R^u{}_{dev} + 3\,\nabla_a R^u{}_{bch} \partial_d \Gamma^h{}_{ve} - 3\,\nabla_a R^h{}_{bcv} \partial_d \Gamma^u{}_{he} + R^u{}_{abh} \partial_{cd} \Gamma^h{}_{ve} - R^h{}_{abv} \partial_{cd} \Gamma^u{}_{he}\right)$$

$$\begin{aligned} A^a A^b A^c A^d A^e A^f R^u{}_{efv,abcd} = A^a A^b A^c A^d A^e A^f \big(&\nabla_{abcd} R^u{}_{efv} + 6\,\nabla_{ab} R^u{}_{cdi} \partial_e \Gamma^i{}_{vf} - 6\,\nabla_{ab} R^i{}_{cdv} \partial_e \Gamma^u{}_{if} + 4\,\nabla_a R^u{}_{bci} \partial_{de} \Gamma^i{}_{vf} - 4\,\nabla_a R^i{}_{bcv} \partial_{de} \Gamma^u{}_{if} + R^u{}_{abi} \partial_{cde} \Gamma^i{}_{vf} \\ &- R^i{}_{abv} \partial_{cde} \Gamma^u{}_{if} + 3\,R^u{}_{abi} \partial_c \Gamma^i{}_{dj} \partial_e \Gamma^j{}_{vf} - 6\,R^i{}_{abj} \partial_c \Gamma^j{}_{dv} \partial_e \Gamma^u{}_{if} + 3\,R^i{}_{abv} \partial_c \Gamma^j{}_{id} \partial_e \Gamma^u{}_{jf}\big) \end{aligned}$$

$$\begin{aligned} A^a A^b A^c A^d A^e A^f A^g R^u{}_{fgv,abcd} = A^a A^b A^c A^d A^e A^f A^g \big(&\nabla_{abcde} R^u{}_{fgv} + 10\,\nabla_{abc} R^u{}_{dej} \partial_f \Gamma^j{}_{vg} - 10\,\nabla_{abc} R^j{}_{dev} \partial_f \Gamma^u{}_{jg} + 10\,\nabla_{ab} R^u{}_{cdj} \partial_{ef} \Gamma^j{}_{vg} - 10\,\nabla_{ab} R^j{}_{cdv} \partial_{ef} \Gamma^u{}_{jg} \\ &+ 5\,\nabla_a R^u{}_{bcj} \partial_{def} \Gamma^j{}_{vg} - 5\,\nabla_a R^j{}_{bcv} \partial_{def} \Gamma^u{}_{jg} + 15\,\nabla_a R^u{}_{bcj} \partial_d \Gamma^j{}_{ek} \partial_f \Gamma^k{}_{vg} - 30\,\nabla_a R^j{}_{bck} \partial_d \Gamma^k{}_{ev} \partial_f \Gamma^u{}_{jg} + 15\,\nabla_a R^j{}_{bcv} \partial_d \Gamma^k{}_{je} \partial_f \Gamma^u{}_{kg} \\ &+ R^u{}_{abj} \partial_{cdef} \Gamma^j{}_{vg} - R^j{}_{abv} \partial_{cdef} \Gamma^u{}_{jg} + 4\,R^u{}_{abj} \partial_f \Gamma^k{}_{vg} \partial_{cd} \Gamma^j{}_{ek} + 6\,R^u{}_{abj} \partial_c \Gamma^j{}_{dk} \partial_{ef} \Gamma^k{}_{vg} - 8\,R^j{}_{abk} \partial_f \Gamma^u{}_{jg} \partial_{cd} \Gamma^k{}_{ev} \\ &- 10\,R^j{}_{abk} \partial_c \Gamma^k{}_{dv} \partial_{ef} \Gamma^u{}_{jg} + 4\,R^j{}_{abv} \partial_f \Gamma^u{}_{kg} \partial_{cd} \Gamma^k{}_{je} + 6\,R^j{}_{abv} \partial_c \Gamma^k{}_{jd} \partial_{ef} \Gamma^u{}_{kg} - 2\,R^j{}_{abk} \partial_c \Gamma^u{}_{jd} \partial_{ef} \Gamma^k{}_{vg}\big) \end{aligned}$$

```
# ==========================================================================================
#   export derivatives for use by metric.cdbp and metric-inv.cdbp
# ==========================================================================================

@distribute!(pderiv01); "pderiv01.del"
@distribute!(pderiv02); "pderiv02.del"
@distribute!(pderiv03); "pderiv03.del"
@distribute!(pderiv04); "pderiv04.del"
@distribute!(pderiv05);
```