Taylor series

The textbook definition of the Taylor series for $f(x)$ aound $x = 0$ is

$$f(x) = f + \frac{x}{1!}\frac{\partial f}{\partial x} + \frac{x^2}{2!}\frac{\partial^2 f}{\partial x^2} + \frac{x^3}{3!}\frac{\partial^3 f}{\partial x^3} + \cdots + \frac{x^n}{n!}\frac{\partial f^n}{\partial x^n} + \cdots$$

where f and all its derivatives on the right hand side are evaluated at $x = 0$.

Here we will make a slight variation. We compute the n-th term by first computing

$$A\frac{\partial}{\partial x}\left(A\frac{\partial}{\partial x}\left(A\frac{\partial}{\partial x}\left(A\frac{\partial}{\partial x}\cdots\left(A\frac{\partial f}{\partial x}\right)\cdots\right)\right)\right)$$

with $0 = \partial A/\partial x$. Then we evaluate the result at $x = 0$ and finally replace A with x.

The value of this small change is that it makes the Cadabra code rather simple. Each term is generated by applying the operator $A\partial/\partial x$ to the previous term.

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{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#,v#}::Indices.

\partial_{#}::PartialDerivative.

# -----------------------------------------------------------------------------------------------
# here we will build the Taylor series for the function $B^{bc}\Gamma^{a}_{bc}(x)$ around $x^{a}=0$.
# -----------------------------------------------------------------------------------------------

function:=B^{b c} \Gamma^{a}_{b c}:

@print["\Gamma^{a}(x)="~@(function)~"(x) ?"];

# --- Taylor series ------------------------------------------------------------------------------

term00:=@(function):

tmp:=A^{a} \partial_{a}{@(term00)}:
@distribute!(tmp):
@prodrule!(tmp):
@distribute!(tmp):
@substitute!(tmp)(\partial_{a}{B^{b c}} -> 0):
term01:=@(tmp):

tmp:=A^{a} \partial_{a}{@(term01)}:
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@distribute!(tmp):
@prodrule!(tmp):
@distribute!(tmp):
@substitute!(tmp)(\partial_{a}{B^{b c}} -> 0, \partial_{a}{A^{b}} -> 0):
term02:=@(tmp):

tmp:=A^{a} \partial_{a}{@(term02)}:
@distribute!(tmp):
@prodrule!(tmp):
@distribute!(tmp):
@substitute!(tmp)(\partial_{a}{B^{b c}} -> 0, \partial_{a}{A^{b}} -> 0):
term03:=@(tmp):

tmp:=A^{a} \partial_{a}{@(term03)}:
@distribute!(tmp):
@prodrule!(tmp):
@distribute!(tmp):
@substitute!(tmp)(\partial_{a}{B^{b c}} -> 0, \partial_{a}{A^{b}} -> 0):
term04:=@(tmp):

# --- now evaluate derivatives at A = 0 -----------------------------------------------------------
# --- we assume that the derivatives, deriv0*, have been computed elsewhere

# @substitute!(term00)(\Gamma^{a}_{b c} -> @(deriv00)):
# @substitute!(term01)(\partial_{d}{\Gamma^{a}_{b c}} -> @(deriv01)):
# @substitute!(term02)(\partial_{d e}{\Gamma^{a}_{b c}} -> @(deriv02)):
# @substitute!(term03)(\partial_{d e f}{\Gamma^{a}_{b c}} -> @(deriv03)):
# @substitute!(term04)(\partial_{d e f g}{\Gamma^{a}_{b c}} -> @(deriv04)):

# --- rebuild the function --------------------------------------------------------------------------

function:=@(term00) + @(term01) + (1/2) @(term02) + (1/6) @(term03) + (1/24) @(term04):
@substitute!(%)(A^{a} -> x^{a}):
@factor_out!!(%){B^{a b}}:

@print["\Gamma^{a}(x)="~@(function)~"?"];
```

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# --- the function and its Taylor series -----------------------------------------------------------
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$$\Gamma^{a}\left(x\right)=B^{bc}\Gamma^{a}{}_{bc}\left(x\right)$$

$$\Gamma^{a}\left(x\right)=B^{bc}\left(\Gamma^{a}{}_{bc}+x^{d}\partial_{d}\Gamma^{a}{}_{bc}+\frac{1}{2}x^{e}x^{d}\partial_{ed}\Gamma^{a}{}_{bc}+\frac{1}{6}x^{f}x^{e}x^{d}\partial_{fed}\Gamma^{a}{}_{bc}+\frac{1}{24}x^{g}x^{f}x^{e}x^{d}\partial_{gfed}\Gamma^{a}{}_{bc}\right)$$