Geodesic IVP

Our game here is to find the solution of

$$0 = \frac{d^2x^a}{ds^2} + \Gamma^a_{bc}(x)\frac{dx^b}{ds}\frac{dx^c}{ds}$$

subject to the initial conditions $x^a(s) = x^a$ and $dx^a(s)/ds = \dot{x}^a$ at $s = 0$

Algorithm

By successive differentiation of the above equation we can compute

$$\frac{d^nx^a}{ds^n} = -\Gamma^a_{bc\underline{d}_n}\frac{dx^b}{ds}\frac{dx^c}{ds}\frac{dx^{\cdot\underline{d}_n}}{ds}$$

at $s = 0$. The $\Gamma^a_{bc\underline{d}_n}$ are the *generalised connections*.

We can then construct the Taylor series solution for $x^a(s)$

$$x^a(s) = x^a - \sum_{k=2}^{\infty}\frac{1}{k!}\Gamma^a_{bc\underline{d}_k}\dot{x}^b\dot{x}^c\dot{x}^{\cdot\underline{d}_k}$$

```
# =========================================================================================
# Compute the generalised connection, for later use in series expansion of x^a(s)
# =========================================================================================

::KeepHistory(false).
::PostDefaultRules( @@collect_terms!(%), @@sumflatten!(%) ).

{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u#,v#}::Indices.

\nabla_{#}::PartialDerivative.
\partial_{#}::PartialDerivative.

g_{a b}::Metric.
g^{a b}::Metric.
\delta^{a}_{b}::KroneckerDelta.
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R_{a b c d}::RiemannTensor.
R^{a}_{b c d}::RiemannTensor.

# --- the Q are shorthand for the genGamma, saves typing ----------------------------------------

{Q^{a}_{b c},Q^{a}_{b c d},Q^{a}_{b c d e},Q^{a}_{b c d e f},Dx^{a}}::SortOrder.

Q^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).
Q^{a}_{b c d}::TableauSymmetry(shape={3}, indices={1,2,3}).
Q^{a}_{b c d e}::TableauSymmetry(shape={4}, indices={1,2,3,4}).
Q^{a}_{b c d e f}::TableauSymmetry(shape={5}, indices={1,2,3,4,5}).
Q^{a}_{b c d e f g}::TableauSymmetry(shape={6}, indices={1,2,3,4,5,6}).

\Gamma^{a}_{b c}::TableauSymmetry(shape={2}, indices={1,2}).

# ==========================================================================================
#   recursively compute $d^n x^a/ds^n$ at $s=0$
# ==========================================================================================

# --- $A^a = x^a$, $B^a = dx^a/ds$ at $s=0$

deriv00:=A^{a};
deriv01:=B^{a};

deriv02:=-\Gamma^{a}_{b c} B^{b} B^{c};

deriv03:=\nabla{@(deriv02)}:
@prodrule!(%):
@substitute!(%)(\nabla{B^{a}}->@(deriv02)):
@substitute!(%)(\nabla{\Gamma^{m}_{s t}}->B^{d}\partial_{d}{\Gamma^{m}_{s t}}):
@prodsort!(%): @rename_dummies!(%): @canonicalise!(%);

deriv04:=\nabla{@(deriv03)}:
@distribute!(%):
@prodrule!(%):
@substitute!(%)(\nabla{B^{a}}->@(deriv02)):
@substitute!(%)(\nabla{\Gamma^{m}_{s t}}->B^{d}\partial_{d}{\Gamma^{m}_{s t}}):
@substitute!(%)(\nabla{\partial_{e}{\Gamma^{m}_{s t}}}->B^{d}\partial_{d e}{\Gamma^{m}_{s t}}):
@prodsort!(%): @rename_dummies!(%): @canonicalise!(%);
```

```
deriv05:=\nabla{@(deriv04)}:
@distribute!(%):
@prodrule!(%):
@substitute!(%)(\nabla{B^{a}}->@(deriv02)):
@substitute!(%)(\nabla{\Gamma^{m}_{s t}}->B^{d}\partial_{d}{\Gamma^{m}_{s t}}):
@substitute!(%)(\nabla{\partial_{e}{\Gamma^{m}_{s t}}}->B^{d}\partial_{d e}{\Gamma^{m}_{s t}}):
@substitute!(%)(\nabla{\partial_{e f}{\Gamma^{m}_{s t}}}->B^{d}\partial_{d e f}{\Gamma^{m}_{s t}}):
@prodsort!(%): @rename_dummies!(%): @canonicalise!(%);
```

--- deriv06 to leading order in s equals $-\Gamma^{a}_{bcdefg}$ which vanishes

```
deriv06:=0;
```

$$deriv00 := A^{a}$$

$$deriv01 := B^{a}$$

$$deriv02 := (-1)\, \Gamma^{a}{}_{bc} B^{b} B^{c}$$

$$deriv03 := -B^{b} B^{c} B^{d} \partial_{b}\Gamma^{a}{}_{cd} + 2\, B^{b} B^{c} B^{d} \Gamma^{a}{}_{be} \Gamma^{e}{}_{cd}$$

$$\begin{aligned} deriv04 := {}& B^{b} B^{c} B^{d} B^{e} \Gamma^{f}{}_{bc} \partial_{f}\Gamma^{a}{}_{de} + 4\, B^{b} B^{c} B^{d} B^{e} \Gamma^{f}{}_{bc} \partial_{d}\Gamma^{a}{}_{ef} - B^{b} B^{c} B^{d} B^{e} \partial_{bc}\Gamma^{a}{}_{de} \\ & - 2\, B^{b} B^{c} B^{d} B^{e} \Gamma^{a}{}_{fg} \Gamma^{f}{}_{bc} \Gamma^{g}{}_{de} - 4\, B^{b} B^{c} B^{d} B^{e} \Gamma^{a}{}_{bf} \Gamma^{f}{}_{cg} \Gamma^{g}{}_{de} + 2\, B^{b} B^{c} B^{d} B^{e} \Gamma^{a}{}_{bf} \partial_{c}\Gamma^{f}{}_{de} \end{aligned}$$

$$\begin{aligned} deriv05 := {}& -2\, B^{b} B^{c} B^{d} B^{e} B^{f} \Gamma^{g}{}_{bc} \Gamma^{h}{}_{dg} \partial_{h}\Gamma^{a}{}_{ef} - 6\, B^{b} B^{c} B^{d} B^{e} B^{f} \Gamma^{g}{}_{bc} \Gamma^{h}{}_{de} \partial_{g}\Gamma^{a}{}_{fh} + B^{b} B^{c} B^{d} B^{e} B^{f} \partial_{g}\Gamma^{a}{}_{bc} \partial_{d}\Gamma^{g}{}_{ef} + 3\, B^{b} B^{c} B^{d} B^{e} B^{f} \Gamma^{g}{}_{bc} \partial_{dg}\Gamma^{a}{}_{ef} \\ & - 12\, B^{b} B^{c} B^{d} B^{e} B^{f} \Gamma^{g}{}_{bc} \Gamma^{h}{}_{dg} \partial_{e}\Gamma^{a}{}_{fh} - 6\, B^{b} B^{c} B^{d} B^{e} B^{f} \Gamma^{g}{}_{bc} \Gamma^{h}{}_{de} \partial_{f}\Gamma^{a}{}_{gh} + 6\, B^{b} B^{c} B^{d} B^{e} B^{f} \partial_{b}\Gamma^{a}{}_{cg} \partial_{d}\Gamma^{g}{}_{ef} \\ & + 6\, B^{b} B^{c} B^{d} B^{e} B^{f} \Gamma^{g}{}_{bc} \partial_{de}\Gamma^{a}{}_{fg} - B^{b} B^{c} B^{d} B^{e} B^{f} \partial_{bcd}\Gamma^{a}{}_{ef} + 12\, B^{b} B^{c} B^{d} B^{e} B^{f} \Gamma^{a}{}_{gh} \Gamma^{g}{}_{bc} \Gamma^{h}{}_{di} \Gamma^{i}{}_{ef} - 6\, B^{b} B^{c} B^{d} B^{e} B^{f} \Gamma^{a}{}_{gh} \Gamma^{g}{}_{bc} \partial_{d}\Gamma^{h}{}_{ef} \\ & + 4\, B^{b} B^{c} B^{d} B^{e} B^{f} \Gamma^{a}{}_{bg} \Gamma^{g}{}_{hi} \Gamma^{h}{}_{cd} \Gamma^{i}{}_{ef} + 8\, B^{b} B^{c} B^{d} B^{e} B^{f} \Gamma^{a}{}_{bg} \Gamma^{g}{}_{ch} \Gamma^{h}{}_{di} \Gamma^{i}{}_{ef} - 8\, B^{b} B^{c} B^{d} B^{e} B^{f} \Gamma^{a}{}_{bg} \Gamma^{h}{}_{cd} \partial_{e}\Gamma^{g}{}_{fh} \\ & - 4\, B^{b} B^{c} B^{d} B^{e} B^{f} \Gamma^{a}{}_{bg} \Gamma^{g}{}_{ch} \partial_{d}\Gamma^{h}{}_{ef} - 2\, B^{b} B^{c} B^{d} B^{e} B^{f} \Gamma^{a}{}_{bg} \Gamma^{h}{}_{cd} \partial_{h}\Gamma^{g}{}_{ef} + 2\, B^{b} B^{c} B^{d} B^{e} B^{f} \Gamma^{a}{}_{bg} \partial_{cd}\Gamma^{g}{}_{ef} \end{aligned}$$

$$deriv06 := 0$$

```
# ===============================================================================================
#
# Now we need to expand deriv0* as a Taylor series around x=0
# These will be used in constructing the general solution of the geodesic equation near x=0, i.e. x^a(s)
# To obtain a series accurate to terms including O(L^6) we will need the following derivatives at x=0
#
#   deriv02,a   deriv02,ab   deriv02,abc   deriv02,abcd
#   deriv03,a   deriv03,ab   deriv03,abc
#   deriv04,a   deriv04,ab
#   deriv05,a
#
# ===============================================================================================

# --- Taylor series for deriv02 -------------------------------------------------------------------

@distribute!(deriv02):
term00:=@(deriv02):

tmp:=A^{a} \partial_{a}{@(term00)}:
@distribute!(tmp): @prodrule!(tmp): @distribute!(tmp):
@substitute!(tmp)(\partial_{a}{B^{b}} -> 0):
term01:=@(tmp):

tmp:=A^{a} \partial_{a}{@(term01)}:
@distribute!(tmp): @prodrule!(tmp): @distribute!(tmp):
@substitute!(tmp)(\partial_{a}{B^{b}} -> 0, \partial_{a}{A^{b}} -> 0):
term02:=@(tmp):

tmp:=A^{a} \partial_{a}{@(term02)}:
@distribute!(tmp): @prodrule!(tmp): @distribute!(tmp):
@substitute!(tmp)(\partial_{a}{B^{b}} -> 0, \partial_{a}{A^{b}} -> 0):
term03:=@(tmp):

tmp:=A^{a} \partial_{a}{@(term03)}:
@distribute!(tmp): @prodrule!(tmp): @distribute!(tmp):
@substitute!(tmp)(\partial_{a}{B^{b}} -> 0, \partial_{a}{A^{b}} -> 0):
term04:=@(tmp):

@substitute!(term00)(\partial_{a b c d}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c d}):
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```
@substitute!(term00)(\partial_{a b c}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c}):
@substitute!(term00)(\partial_{a b}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b}):
@substitute!(term00)(\partial_{a}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a}):
@substitute!(term00)(\Gamma^{p}_{m n} -> Q^{p}_{m n}):

@substitute!(term01)(\partial_{a b c d}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c d}):
@substitute!(term01)(\partial_{a b c}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c}):
@substitute!(term01)(\partial_{a b}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b}):
@substitute!(term01)(\partial_{a}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a}):
@substitute!(term01)(\Gamma^{p}_{m n} -> Q^{p}_{m n}):

@substitute!(term02)(\partial_{a b c d}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c d}):
@substitute!(term02)(\partial_{a b c}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c}):
@substitute!(term02)(\partial_{a b}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b}):
@substitute!(term02)(\partial_{a}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a}):
@substitute!(term02)(\Gamma^{p}_{m n} -> Q^{p}_{m n}):

@substitute!(term03)(\partial_{a b c d}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c d}):
@substitute!(term03)(\partial_{a b c}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c}):
@substitute!(term03)(\partial_{a b}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b}):
@substitute!(term03)(\partial_{a}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a}):
@substitute!(term03)(\Gamma^{p}_{m n} -> Q^{p}_{m n}):

@substitute!(term04)(\partial_{a b c d}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c d}):
@substitute!(term04)(\partial_{a b c}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c}):
@substitute!(term04)(\partial_{a b}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b}):
@substitute!(term04)(\partial_{a}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a}):
@substitute!(term04)(\Gamma^{p}_{m n} -> Q^{p}_{m n}):

# --- rebuild deriv02

deriv02:=@(term00) + @(term01) + (1/2) @(term02) + (1/6) @(term03) + (1/24) @(term04):

# --- impose RNC conditions

@substitute!(deriv02)(Q^{p}_{m n} -> 0):
@substitute!(deriv02)(B^a B^b Q^{p}_{a b} -> 0);

# --- Taylor series for deriv03 -------------------------------------------------------------------

@distribute!(deriv03):
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```
term00:=@(deriv03):

tmp:=A^{a} \partial_{a}{@(term00)}:
@distribute!(tmp): @prodrule!(tmp): @distribute!(tmp):
@substitute!(tmp)(\partial_{a}{B^{b}} -> 0):
term01:=@(tmp):

tmp:=A^{a} \partial_{a}{@(term01)}:
@distribute!(tmp): @prodrule!(tmp): @distribute!(tmp):
@substitute!(tmp)(\partial_{a}{B^{b}} -> 0, \partial_{a}{A^{b}} -> 0):
term02:=@(tmp):

tmp:=A^{a} \partial_{a}{@(term02)}:
@distribute!(tmp): @prodrule!(tmp): @distribute!(tmp):
@substitute!(tmp)(\partial_{a}{B^{b}} -> 0, \partial_{a}{A^{b}} -> 0):
term03:=@(tmp):

@substitute!(term00)(\partial_{a b c}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c}):
@substitute!(term00)(\partial_{a b}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b}):
@substitute!(term00)(\partial_{a}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a}):
@substitute!(term00)(\Gamma^{p}_{m n} -> Q^{p}_{m n}):

@substitute!(term01)(\partial_{a b c}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c}):
@substitute!(term01)(\partial_{a b}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b}):
@substitute!(term01)(\partial_{a}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a}):
@substitute!(term01)(\Gamma^{p}_{m n} -> Q^{p}_{m n}):

@substitute!(term02)(\partial_{a b c}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c}):
@substitute!(term02)(\partial_{a b}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b}):
@substitute!(term02)(\partial_{a}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a}):
@substitute!(term02)(\Gamma^{p}_{m n} -> Q^{p}_{m n}):

@substitute!(term03)(\partial_{a b c}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c}):
@substitute!(term03)(\partial_{a b}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b}):
@substitute!(term03)(\partial_{a}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a}):
@substitute!(term03)(\Gamma^{p}_{m n} -> Q^{p}_{m n}):

# --- rebuild deriv03
```

```
deriv03:=@(term00) + @(term01) + (1/2) @(term02) + (1/6) @(term03):

# --- impose RNC conditions

@substitute!(deriv03)(Q^{p}_{m n} -> 0):
@substitute!(deriv03)(B^a B^b B^c Q^{p}_{a b c} -> 0);

# --- Taylor series for deriv04 ------------------------------------------------------------------

@distribute!(deriv04):
term00:=@(deriv04):

tmp:=A^{a} \partial_{a}{@(term00)}:
@distribute!(tmp): @prodrule!(tmp): @distribute!(tmp):
@substitute!(tmp)(\partial_{a}{B^{b}} -> 0):
term01:=@(tmp):

tmp:=A^{a} \partial_{a}{@(term01)}:
@distribute!(tmp): @prodrule!(tmp): @distribute!(tmp):
@substitute!(tmp)(\partial_{a}{B^{b}} -> 0, \partial_{a}{A^{b}} -> 0):
term02:=@(tmp):

@substitute!(term00)(\partial_{a b}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b}):
@substitute!(term00)(\partial_{a}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a}):
@substitute!(term00)(\Gamma^{p}_{m n} -> Q^{p}_{m n}):

@substitute!(term01)(\partial_{a b}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b}):
@substitute!(term01)(\partial_{a}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a}):
@substitute!(term01)(\Gamma^{p}_{m n} -> Q^{p}_{m n}):

@substitute!(term02)(\partial_{a b}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b}):
@substitute!(term02)(\partial_{a}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a}):
@substitute!(term02)(\Gamma^{p}_{m n} -> Q^{p}_{m n}):

# --- rebuild deriv04

deriv04:=@(term00) + @(term01) + (1/2) @(term02):

# --- impose RNC conditions

@substitute!(deriv04)(Q^{p}_{m n} -> 0):
@substitute!(deriv04)(B^a B^b B^c Q^{p}_{a b c} -> 0):
```

```
@substitute!(deriv04)(B^a B^b B^c B^d Q^{p}_{a b c d} -> 0);

# --- Taylor series for deriv05 ---------------------------------------------------------------

@distribute!(deriv05):
term00:=@(deriv05):

tmp:=A^{a} \partial_{a}{@(term00)}:
@distribute!(tmp): @prodrule!(tmp): @distribute!(tmp):
@substitute!(tmp)(\partial_{a}{B^{b}} -> 0):
term01:=@(tmp):

@substitute!(term00)(\partial_{a b c}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c}):
@substitute!(term00)(\partial_{a b}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b}):
@substitute!(term00)(\partial_{a}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a}):
@substitute!(term00)(\Gamma^{p}_{m n} -> Q^{p}_{m n}):

@substitute!(term01)(\partial_{a b c}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b c}):
@substitute!(term01)(\partial_{a b}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a b}):
@substitute!(term01)(\partial_{a}{\Gamma^{p}_{m n}} -> Q^{p}_{m n a}):
@substitute!(term01)(\Gamma^{p}_{m n} -> Q^{p}_{m n}):

# --- rebuild deriv05

deriv05:=@(term00) + @(term01):

# --- impose RNC conditions

@substitute!(deriv05)(Q^{p}_{m n} -> 0):
@substitute!(deriv05)(B^a B^b B^c Q^{p}_{a b c} -> 0):
@substitute!(deriv05)(B^a B^b B^c B^d Q^{p}_{a b c d} -> 0):
@substitute!(deriv05)(B^a B^b B^c B^d B^e Q^{p}_{a b c d e} -> 0);
```

$$deriv02 := -A^d Q^a{}_{bcd} B^b B^c - \frac{1}{2} A^e A^d Q^a{}_{bced} B^b B^c - \frac{1}{6} A^f A^e A^d Q^a{}_{bcfed} B^b B^c - \frac{1}{24} A^g A^f A^e A^d Q^a{}_{bcgfed} B^b B^c$$

$$\begin{aligned} deriv03 := &-A^f B^b B^c B^d Q^a{}_{cdfb} - \frac{1}{2} A^g A^f B^b B^c B^d Q^a{}_{cdgfb} + A^g A^f B^b B^c B^d Q^a{}_{bef} Q^e{}_{cdg} + A^g A^f B^b B^c B^d Q^a{}_{beg} Q^e{}_{cdf} \\ &+ \frac{1}{3} A^h A^g A^f B^b B^c B^d Q^a{}_{begf} Q^e{}_{cdh} + \frac{1}{3} A^h A^g A^f B^b B^c B^d Q^a{}_{behf} Q^e{}_{cdg} + \frac{1}{3} A^h A^g A^f B^b B^c B^d Q^a{}_{bef} Q^e{}_{cdhg} \\ &+ \frac{1}{3} A^h A^g A^f B^b B^c B^d Q^a{}_{behg} Q^e{}_{cdf} + \frac{1}{3} A^h A^g A^f B^b B^c B^d Q^a{}_{beg} Q^e{}_{cdhf} + \frac{1}{3} A^h A^g A^f B^b B^c B^d Q^a{}_{beh} Q^e{}_{cdgf} \end{aligned}$$

$$deriv04 := A^h B^b B^c B^d B^e Q^f{}_{bch} Q^a{}_{def} + 4\, A^h B^b B^c B^d B^e Q^f{}_{bch} Q^a{}_{efd} + \frac{1}{2}\, A^i A^h B^b B^c B^d B^e Q^f{}_{bcih} Q^a{}_{def} + \frac{1}{2}\, A^i A^h B^b B^c B^d B^e Q^f{}_{bch} Q^a{}_{deif}$$
$$+ \frac{1}{2}\, A^i A^h B^b B^c B^d B^e Q^f{}_{bci} Q^a{}_{dehf} + 2\, A^i A^h B^b B^c B^d B^e Q^f{}_{bcih} Q^a{}_{efd} + 2\, A^i A^h B^b B^c B^d B^e Q^f{}_{bch} Q^a{}_{efid}$$
$$+ 2\, A^i A^h B^b B^c B^d B^e Q^f{}_{bci} Q^a{}_{efhd} + A^i A^h B^b B^c B^d B^e Q^a{}_{bfh} Q^f{}_{deic} + A^i A^h B^b B^c B^d B^e Q^a{}_{bfi} Q^f{}_{dehc}$$

$$deriv05 := A^j B^b B^c B^d B^e B^f Q^a{}_{bcg} Q^g{}_{efjd} + 3\, A^j B^b B^c B^d B^e B^f Q^g{}_{bcj} Q^a{}_{efdg} + 6\, A^j B^b B^c B^d B^e B^f Q^a{}_{cgb} Q^g{}_{efjd} + 6\, A^j B^b B^c B^d B^e B^f Q^g{}_{bcj} Q^a{}_{fgde}$$

```
# --- print the results ---------------------------------------------------------------------------
@print["x^a="~@(deriv00)];
@print["\frac{dx^a}{ds}="~@(deriv01)];
@print["\frac{d^2x^a}{ds^2}="~@(deriv02)~"+\BigO{\eps^6}"];
@print["\frac{d^3x^a}{ds^3}="~@(deriv03)~"+\BigO{\eps^6}"];
@print["\frac{d^4x^a}{ds^4}="~@(deriv04)~"+\BigO{\eps^6}"];
@print["\frac{d^5x^a}{ds^5}="~@(deriv05)~"+\BigO{\eps^6}"];
@print["\frac{d^6x^a}{ds^6}="~@(deriv06)~"+\BigO{\eps^6}"];
```

$$x^a = A^a$$

$$\frac{dx^a}{ds} = B^a$$

$$\frac{d^2x^a}{ds^2} = \left(-A^dQ^a{}_{bcd}B^bB^c - \frac{1}{2}\,A^eA^dQ^a{}_{bced}B^bB^c - \frac{1}{6}\,A^fA^eA^dQ^a{}_{bcfed}B^bB^c - \frac{1}{24}\,A^gA^fA^eA^dQ^a{}_{bcgfed}B^bB^c\right) + \mathcal{O}\left(\epsilon^6\right)$$

$$\begin{aligned}\frac{d^3x^a}{ds^3} = \Big(&-A^fB^bB^cB^dQ^a{}_{cdfb} - \frac{1}{2}\,A^gA^fB^bB^cB^dQ^a{}_{cdgfb} + A^gA^fB^bB^cB^dQ^a{}_{bef}Q^e{}_{cdg} + A^gA^fB^bB^cB^dQ^a{}_{beg}Q^e{}_{cdf} + \frac{1}{3}\,A^hA^gA^fB^bB^cB^dQ^a{}_{begf}Q^e{}_{cdh}\\ &+ \frac{1}{3}\,A^hA^gA^fB^bB^cB^dQ^a{}_{behf}Q^e{}_{cdg} + \frac{1}{3}\,A^hA^gA^fB^bB^cB^dQ^a{}_{bef}Q^e{}_{cdhg} + \frac{1}{3}\,A^hA^gA^fB^bB^cB^dQ^a{}_{behg}Q^e{}_{cdf} + \frac{1}{3}\,A^hA^gA^fB^bB^cB^dQ^a{}_{beg}Q^e{}_{cdhf}\\ &+ \frac{1}{3}\,A^hA^gA^fB^bB^cB^dQ^a{}_{beh}Q^e{}_{cdgf}\Big) + \mathcal{O}\left(\epsilon^6\right)\end{aligned}$$

$$\begin{aligned}\frac{d^4x^a}{ds^4} = \Big(&A^hB^bB^cB^dB^eQ^f{}_{bch}Q^a{}_{def} + 4\,A^hB^bB^cB^dB^eQ^f{}_{bch}Q^a{}_{efd} + \frac{1}{2}\,A^iA^hB^bB^cB^dB^eQ^f{}_{bcih}Q^a{}_{def} + \frac{1}{2}\,A^iA^hB^bB^cB^dB^eQ^f{}_{bch}Q^a{}_{deif}\\ &+ \frac{1}{2}\,A^iA^hB^bB^cB^dB^eQ^f{}_{bci}Q^a{}_{dehf} + 2\,A^iA^hB^bB^cB^dB^eQ^f{}_{bcih}Q^a{}_{efd} + 2\,A^iA^hB^bB^cB^dB^eQ^f{}_{bch}Q^a{}_{efid} + 2\,A^iA^hB^bB^cB^dB^eQ^f{}_{bci}Q^a{}_{efhd}\\ &+ A^iA^hB^bB^cB^dB^eQ^a{}_{bfh}Q^f{}_{deic} + A^iA^hB^bB^cB^dB^eQ^a{}_{bfi}Q^f{}_{dehc}\Big) + \mathcal{O}\left(\epsilon^6\right)\end{aligned}$$

$$\frac{d^5x^a}{ds^5} = \left(A^jB^bB^cB^dB^eB^fQ^a{}_{bcg}Q^g{}_{efjd} + 3\,A^jB^bB^cB^dB^eB^fQ^g{}_{bcj}Q^a{}_{efdg} + 6\,A^jB^bB^cB^dB^eB^fQ^a{}_{cgb}Q^g{}_{efjd} + 6\,A^jB^bB^cB^dB^eB^fQ^g{}_{bcj}Q^a{}_{fgde}\right) + \mathcal{O}\left(\epsilon^6\right)$$

$$\frac{d^6x^a}{ds^6} = 0 + \mathcal{O}\left(\epsilon^6\right)$$

```
# --- imported from connection.lib ------------------------------------------------------------

# gamma(x) = Gamma00 + x^d Gamma01 + x^d x^e Gamma02 + x^d x^e x^f Gamma03 + ...

# Gamma01 = 1! gamma_{,d}     at x=0, symmetrised over {d}
# Gamma02 = 2! gamma_{,de}    at x=0, symmetrised over {d,e} = ( de + ed )/2!
# Gamma03 = 3! gamma_{,def}   at x=0, symmetrised over {d,e,f} = ( def + edf + dfe + ...)/3!

Gamma01:="import connection.lib Gamma01":
@run(Gamma01){"/Users/leo/local/sh/cdbfile"}:

Gamma02:="import connection.lib Gamma02":
@run(Gamma02){"/Users/leo/local/sh/cdbfile"}:

Gamma03:="import connection.lib Gamma03":
@run(Gamma03){"/Users/leo/local/sh/cdbfile"}:

Gamma04:="import connection.lib Gamma04":
@run(Gamma04){"/Users/leo/local/sh/cdbfile"}:

# --- Gamma0* as imported from connection.lib was *not* symmetrised, do so now --------------------

@sym!(Gamma02){ _{d}, _{e} }: @canonicalise!(%):
@sym!(Gamma03){ _{d}, _{e}, _{f} }: @canonicalise!(%):
@sym!(Gamma04){ _{d}, _{e}, _{f}, _{g} }: @canonicalise!(%):

# --- cancel the 1/n! that comes from @sym(...) ---------------------------------------------------

tmp:=@(Gamma02): Gamma02:=  2 @(tmp):
tmp:=@(Gamma03): Gamma03:=  6 @(tmp):
tmp:=@(Gamma04): Gamma04:= 24 @(tmp):

# --- substitute into the deriv0* -----------------------------------------------------------------

@substitute!(deriv02)(Q^{a}{}_{b c d} -> @(Gamma01)):
@substitute!(deriv02)(Q^{a}{}_{b c d e} -> @(Gamma02)):
@substitute!(deriv02)(Q^{a}{}_{b c d e f} -> @(Gamma03)):
@substitute!(deriv02)(Q^{a}{}_{b c d e f g} -> @(Gamma04)):
@distribute!(%): @prodsort!(%): @rename_dummies!(%): @canonicalise!(%):

@substitute!(deriv03)(Q^{a}{}_{b c d} -> @(Gamma01)):
@substitute!(deriv03)(Q^{a}{}_{b c d e} -> @(Gamma02)):
```

```
@substitute!(deriv03)(Q^{a}{}_{b c d e f} -> @(Gamma03)):
@distribute!(%): @prodsort!(%): @rename_dummies!(%): @canonicalise!(%):

@substitute!(deriv04)(Q^{a}{}_{b c d} -> @(Gamma01)):
@substitute!(deriv04)(Q^{a}{}_{b c d e} -> @(Gamma02)):
@substitute!(deriv04)(Q^{a}{}_{b c d e f} -> @(Gamma03)):
@distribute!(%): @prodsort!(%): @rename_dummies!(%): @canonicalise!(%):

@substitute!(deriv05)(Q^{a}{}_{b c d} -> @(Gamma01)):
@substitute!(deriv05)(Q^{a}{}_{b c d e} -> @(Gamma02)):
@substitute!(deriv05)(Q^{a}{}_{b c d e f} -> @(Gamma03)):
@distribute!(%): @prodsort!(%): @rename_dummies!(%): @canonicalise!(%):

@substitute!(deriv06)(Q^{a}{}_{b c d} -> @(Gamma01)):
@substitute!(deriv06)(Q^{a}{}_{b c d e} -> @(Gamma02)):
@substitute!(deriv06)(Q^{a}{}_{b c d e f} -> @(Gamma03)):
@distribute!(%): @prodsort!(%): @rename_dummies!(%): @canonicalise!(%):

# ===============================================================================================
#   force all indices on R to be downstairs
# ===============================================================================================

@substitute!(deriv02)(R^{a}_{b c d} -> R_{a b c d},
                      R_{a}^{b}_{c d} -> R_{a b c d},
                      R^{a}_{b c}^{d} -> R_{a b c d},
                      R_{a b}^{c}_{d} -> R_{a b c d},
                      R_{a b c}^{d} -> R_{a b c d},
                      R^{a}_{b}^{c d} -> R_{a b c d},
                      R^{a}_{b}^{c}_{d} -> R_{a b c d},
                      \nabla^{a}{R_{b c d e}} -> \nabla_{a}{R_{b c d e}},
                      \nabla^{a}_{f}{R_{b c d e}} -> \nabla_{a f}{R_{b c d e}},
                      \nabla^{a}_{f g}{R_{b c d e}} -> \nabla_{a f g}{R_{b c d e}}):

@substitute!(deriv03)(R^{a}_{b c d} -> R_{a b c d},
                      R_{a}^{b}_{c d} -> R_{a b c d},
                      R^{a}_{b c}^{d} -> R_{a b c d},
                      R_{a b}^{c}_{d} -> R_{a b c d},
                      R_{a b c}^{d} -> R_{a b c d},
                      R^{a}_{b}^{c d} -> R_{a b c d},
```

```
                      R^{a}_{b}^{c}_{d} -> R_{a b c d},
                      \nabla^{a}{R_{b c d e}} -> \nabla_{a}{R_{b c d e}},
                      \nabla^{a}_{f}{R_{b c d e}} -> \nabla_{a f}{R_{b c d e}},
                      \nabla^{a}_{f g}{R_{b c d e}} -> \nabla_{a f g}{R_{b c d e}}):

@substitute!(deriv04)(R^{a}_{b c d} -> R_{a b c d},
                      R_{a}^{b}_{c d} -> R_{a b c d},
                      R^{a}_{b c}^{d} -> R_{a b c d},
                      R_{a b}^{c}_{d} -> R_{a b c d},
                      R_{a b c}^{d} -> R_{a b c d},
                      R^{a}_{b}^{c d} -> R_{a b c d},
                      R^{a}_{b}^{c}_{d} -> R_{a b c d},
                      \nabla^{a}{R_{b c d e}} -> \nabla_{a}{R_{b c d e}},
                      \nabla^{a}_{f}{R_{b c d e}} -> \nabla_{a f}{R_{b c d e}},
                      \nabla^{a}_{f g}{R_{b c d e}} -> \nabla_{a f g}{R_{b c d e}}):

@substitute!(deriv05)(R^{a}_{b c d} -> R_{a b c d},
                      R_{a}^{b}_{c d} -> R_{a b c d},
                      R^{a}_{b c}^{d} -> R_{a b c d},
                      R_{a b}^{c}_{d} -> R_{a b c d},
                      R_{a b c}^{d} -> R_{a b c d},
                      R^{a}_{b}^{c d} -> R_{a b c d},
                      R^{a}_{b}^{c}_{d} -> R_{a b c d},
                      \nabla^{a}{R_{b c d e}} -> \nabla_{a}{R_{b c d e}},
                      \nabla^{a}_{f}{R_{b c d e}} -> \nabla_{a f}{R_{b c d e}},
                      \nabla^{a}_{f g}{R_{b c d e}} -> \nabla_{a f g}{R_{b c d e}}):

@substitute!(deriv06)(R^{a}_{b c d} -> R_{a b c d},
                      R_{a}^{b}_{c d} -> R_{a b c d},
                      R^{a}_{b c}^{d} -> R_{a b c d},
                      R_{a b}^{c}_{d} -> R_{a b c d},
                      R_{a b c}^{d} -> R_{a b c d},
                      R^{a}_{b}^{c d} -> R_{a b c d},
                      R^{a}_{b}^{c}_{d} -> R_{a b c d},
                      \nabla^{a}{R_{b c d e}} -> \nabla_{a}{R_{b c d e}},
                      \nabla^{a}_{f}{R_{b c d e}} -> \nabla_{a f}{R_{b c d e}},
                      \nabla^{a}_{f g}{R_{b c d e}} -> \nabla_{a f g}{R_{b c d e}}):
```

```
# --- compute the generalised connections ----------------------------------------------------------

{B^{a},x^{a},R_{a b c d},R^{a}_{b c d},\nabla_{a}{R_{b c d e}}}::SortOrder.

@prodsort!(deriv02):
@prodsort!(deriv03):
@prodsort!(deriv04):
@prodsort!(deriv05):
@prodsort!(deriv06):

@rename_dummies!(deriv02):
@rename_dummies!(deriv03):
@rename_dummies!(deriv04):
@rename_dummies!(deriv05):
@rename_dummies!(deriv06):

@canonicalise!(deriv02):
@canonicalise!(deriv03):
@canonicalise!(deriv04):
@canonicalise!(deriv05):
@canonicalise!(deriv06):

# --- raise index {a} and set A^a -> x^a

genGamma02:= - @(deriv02)) g^{a u1}:
@distribute!(%): @eliminate_metric!(%):
tmp:=@(genGamma02) \delta^{a}_{u1}: @distribute!(%): @eliminate_kr!(%):
@substitute!(%)(A^{a}->x^{a}): @prodsort!(%): @rename_dummies!(%): @factor_out!!(%)(B^{a}):
genGamma02:= @(tmp); "genGamma02.del"

genGamma03:= - @(deriv03)) g^{a u1}:
@distribute!(%): @eliminate_metric!(%):
tmp:=@(genGamma03) \delta^{a}_{u1}: @distribute!(%): @eliminate_kr!(%):
@substitute!(%)(A^{a}->x^{a}): @prodsort!(%): @rename_dummies!(%): @factor_out!!(%)(B^{a}):
genGamma03:= @(tmp); "genGamma03.del"

genGamma04:= - @(deriv04) g^{a u1}:
@distribute!(%): @eliminate_metric!(%):
tmp:=@(genGamma04) \delta^{a}_{u1}: @distribute!(%): @eliminate_kr!(%):
```

```
@substitute!(%)(A^{a}->x^{a}): @prodsort!(%): @rename_dummies!(%): @factor_out!!(%)(B^{a}):
genGamma04:= @(tmp); "genGamma04.del"

genGamma05:= - @(deriv05)) g^{a u1}:
@distribute!(%): @eliminate_metric!(%):
tmp:=@(genGamma05) \delta^{a}_{u1}: @distribute!(%): @eliminate_kr!(%):
@substitute!(%)(A^{a}->x^{a}): @prodsort!(%): @rename_dummies!(%): @factor_out!!(%)(B^{a}):
genGamma05:= @(tmp); "genGamma05.del"

genGamma06:= - @(deriv06)) g^{a u1}:
@distribute!(%): @eliminate_metric!(%):
tmp:=@(genGamma06) \delta^{a}_{u1}: @distribute!(%): @eliminate_kr!(%):
@substitute!(%)(A^{a}->x^{a}): @prodsort!(%): @rename_dummies!(%): @factor_out!!(%)(B^{a}):
genGamma06:= @(tmp); "genGamma06.del"
```

```
# --- the generalised connections ------------------------------------------------------------------
@print["\genGammaA="~@(genGamma02)];
@print["\genGammaB="~@(genGamma03)];
@print["\genGammaC="~@(genGamma04)];
@print["\genGammaD="~@(genGamma05)];
@print["\genGammaE="~@(genGamma06)];
```

$$
\begin{aligned}
B^bB^c\Gamma^a_{bc}(x) = B^bB^c\Big(&\frac{2}{3}\,x^dR^a{}_{bdc}+\frac{1}{6}\,x^dx^e\nabla_bR^a{}_{dec}+\frac{1}{3}\,x^dx^e\nabla_dR^a{}_{bec}+\frac{1}{12}\,x^dx^e\nabla^aR_{dbec}+\frac{8}{45}\,x^dx^ex^fR^a{}_{deg}R_{fbcg}-\frac{4}{45}\,x^dx^ex^fR^a{}_{bdg}R_{ecfg}-\frac{2}{45}\,x^dx^ex^fR^a{}_{dbg}R_{ecfg}\\
&+\frac{1}{10}\,x^dx^ex^f\nabla_{db}R^a{}_{efc}+\frac{1}{10}\,x^dx^ex^f\nabla_{de}R^a{}_{bfc}-\frac{2}{45}\,x^dx^ex^fR^a{}_{gdb}R_{ecfg}+\frac{1}{20}\,x^dx^ex^f\nabla^a{}_dR_{ebfc}+\frac{4}{45}\,x^dx^ex^fx^gR_{dbch}\nabla_eR^a{}_{fgh}\\
&+\frac{1}{30}\,x^dx^ex^fx^gR^a{}_{deh}\nabla_bR_{fcgh}+\frac{4}{45}\,x^dx^ex^fx^gR^a{}_{deh}\nabla_fR_{gbch}-\frac{2}{45}\,x^dx^ex^fx^gR^a{}_{bdh}\nabla_eR_{fcgh}-\frac{1}{45}\,x^dx^ex^fx^gR^a{}_{dbh}\nabla_eR_{fcgh}-\frac{1}{45}\,x^dx^ex^fx^gR_{dbeh}\nabla_cR^a{}_{fgh}\\
&-\frac{2}{45}\,x^dx^ex^fx^gR_{dbeh}\nabla_fR^a{}_{cgh}-\frac{1}{45}\,x^dx^ex^fx^gR_{dbeh}\nabla_fR^a{}_{gch}+\frac{1}{30}\,x^dx^ex^fx^g\nabla_{deb}R^a{}_{fgc}+\frac{1}{45}\,x^dx^ex^fx^g\nabla_{def}R^a{}_{bgc}-\frac{1}{36}\,x^dx^ex^fx^gR^a{}_{deh}\nabla_hR_{fbgc}\\
&-\frac{1}{45}\,x^dx^ex^fx^gR^a{}_{hdb}\nabla_eR_{fcgh}-\frac{1}{45}\,x^dx^ex^fx^gR_{dbeh}\nabla^aR_{fcgh}-\frac{1}{45}\,x^dx^ex^fx^gR_{dbeh}\nabla_fR^a{}_{hgc}+\frac{1}{60}\,x^dx^ex^fx^g\nabla^a{}_{de}R_{fbgc}\Big)
\end{aligned}
$$

$$
\begin{aligned}
B^bB^cB^d\Gamma^a_{bcd}(x) = B^bB^cB^d\Big(&\frac{1}{2}\,x^e\nabla_bR^a{}_{ced}+\frac{2}{15}\,x^ex^fR^a{}_{ebg}R_{fcdg}+\frac{8}{15}\,x^ex^fR^a{}_{beg}R_{fcdg}-\frac{2}{15}\,x^ex^fR^a{}_{bcg}R_{edfg}+\frac{3}{10}\,x^ex^f\nabla_{eb}R^a{}_{cfd}+\frac{1}{10}\,x^ex^f\nabla_{bc}R^a{}_{efd}\\
&+\frac{2}{5}\,x^ex^fR^a{}_{geb}R_{fcdg}+\frac{1}{20}\,x^ex^f\nabla^a{}_bR_{ecfd}+\frac{1}{9}\,x^ex^fx^gR_{ebch}\nabla_dR^a{}_{fgh}+\frac{1}{9}\,x^ex^fx^gR_{ebch}\nabla_hR^a{}_{fgd}+\frac{2}{9}\,x^ex^fx^gR_{ebch}\nabla_fR^a{}_{dgh}+\frac{2}{9}\,x^ex^fx^gR_{ebch}\nabla_fR^a{}_{hgd}\\
&+\frac{1}{9}\,x^ex^fx^gR_{ebch}\nabla^aR_{fdgh}+\frac{1}{9}\,x^ex^fx^gR^a{}_{beh}\nabla_cR_{fdgh}+\frac{2}{9}\,x^ex^fx^gR^a{}_{beh}\nabla_fR_{gcdh}-\frac{1}{18}\,x^ex^fx^gR^a{}_{beh}\nabla_hR_{fcgd}+\frac{1}{9}\,x^ex^fx^gR^a{}_{heb}\nabla_cR_{fdgh}\\
&+\frac{2}{9}\,x^ex^fx^gR^a{}_{heb}\nabla_fR_{gcdh}-\frac{1}{18}\,x^ex^fx^gR^a{}_{heb}\nabla_hR_{fcgd}\Big)
\end{aligned}
$$

$$
\begin{aligned}
B^bB^cB^dB^e\Gamma^a_{bcde}(x) = B^bB^cB^dB^e\Big(&\frac{4}{9}\,x^fR^a{}_{bcg}R_{fdeg}+\frac{1}{9}\,x^fx^gR^a{}_{bch}\nabla_dR_{fegh}+\frac{2}{9}\,x^fx^gR^a{}_{bch}\nabla_fR_{gdeh}-\frac{1}{18}\,x^fx^gR^a{}_{bch}\nabla_hR_{fdge}+\frac{1}{9}\,x^fx^gR_{fbch}\nabla_dR^a{}_{geh}\\
&+\frac{5}{9}\,x^fx^gR_{fbch}\nabla_dR^a{}_{hge}+\frac{2}{9}\,x^fx^gR_{fbch}\nabla_gR^a{}_{deh}+\frac{4}{9}\,x^fx^gR_{fbch}\nabla_hR^a{}_{dge}+\frac{1}{9}\,x^fx^gR_{fbch}\nabla^aR_{gdeh}+\frac{2}{3}\,x^fx^gR_{fbch}\nabla_dR^a{}_{egh}\\
&+\frac{1}{3}\,x^fx^gR^a{}_{bfh}\nabla_cR_{gdeh}+\frac{1}{3}\,x^fx^gR^a{}_{hfb}\nabla_cR_{gdeh}\Big)
\end{aligned}
$$

$$B^b B^c B^d B^e B^f \Gamma^a_{bcdef}(x) = B^b B^c B^d B^e B^f \left(\frac{2}{3}\, x^g R^a{}_{bch} \nabla_d R_{gefh} + x^g R_{gbch} \nabla_d R^a{}_{efh} \right)$$

$$B^b B^c B^d B^e B^f B^g \Gamma^a_{bcdefg}(x) = 0$$

```
# --- terms in the series solution of the gedodesic ivp ------------------------------------------

deriv02:= - @(genGamma02): @substitute!(deriv02)(x^{a}->A^{a}):
deriv03:= - @(genGamma03): @substitute!(deriv03)(x^{a}->A^{a}):
deriv04:= - @(genGamma04): @substitute!(deriv04)(x^{a}->A^{a}):
deriv05:= - @(genGamma05): @substitute!(deriv05)(x^{a}->A^{a}):
deriv06:= - @(genGamma06): @substitute!(deriv06)(x^{a}->A^{a}):

@print["x^a="~@(deriv00)];
@print["\frac{dx^a}{ds}="~@(deriv01)];
@print["\frac{d^2x^a}{ds^2}="~@(deriv02)~"+\BigO{\eps^6}"];
@print["\frac{d^3x^a}{ds^3}="~@(deriv03)~"+\BigO{\eps^6}"];
@print["\frac{d^4x^a}{ds^4}="~@(deriv04)~"+\BigO{\eps^6}"];
@print["\frac{d^5x^a}{ds^5}="~@(deriv05)~"+\BigO{\eps^6}"];
@print["\frac{d^6x^a}{ds^6}="~@(deriv06)~"+\BigO{\eps^6}"];
```

$$x^a = A^a$$

$$\frac{dx^a}{ds} = B^a$$

$$\begin{aligned}\frac{d^2x^a}{ds^2} = (-1)\ B^b B^c \Bigg(&\frac{2}{3}\,A^d R^a{}_{bdc} + \frac{1}{6}\,A^d A^e \nabla_b R^a{}_{dec} + \frac{1}{3}\,A^d A^e \nabla_d R^a{}_{bec} + \frac{1}{12}\,A^d A^e \nabla^a R_{dbec} + \frac{8}{45}\,A^d A^e A^f R^a{}_{deg} R_{fbcg} - \frac{4}{45}\,A^d A^e A^f R^a{}_{bdg} R_{ecfg} - \frac{2}{45}\,A^d A^e A^f R^a{}_{dbg} R_{ecfg}\\
&+ \frac{1}{10}\,A^d A^e A^f \nabla_{db} R^a{}_{efc} + \frac{1}{10}\,A^d A^e A^f \nabla_{de} R^a{}_{bfc} - \frac{2}{45}\,A^d A^e A^f R^a{}_{gdb} R_{ecfg} + \frac{1}{20}\,A^d A^e A^f \nabla^a{}_d R_{ebfc} + \frac{4}{45}\,A^d A^e A^f A^g R_{dbch} \nabla_e R^a{}_{fgh}\\
&+ \frac{1}{30}\,A^d A^e A^f A^g R^a{}_{deh} \nabla_b R_{fcgh} + \frac{4}{45}\,A^d A^e A^f A^g R^a{}_{deh} \nabla_f R_{gbch} - \frac{2}{45}\,A^d A^e A^f A^g R^a{}_{bdh} \nabla_e R_{fcgh} - \frac{1}{45}\,A^d A^e A^f A^g R^a{}_{dbh} \nabla_e R_{fcgh} - \frac{1}{45}\,A^d A^e A^f A^g R_{dbeh} \nabla_c R^a{}_{fgh}\\
&- \frac{2}{45}\,A^d A^e A^f A^g R_{dbeh} \nabla_f R^a{}_{cgh} - \frac{1}{45}\,A^d A^e A^f A^g R_{dbeh} \nabla_f R^a{}_{gch} + \frac{1}{30}\,A^d A^e A^f A^g \nabla_{deb} R^a{}_{fgc} + \frac{1}{45}\,A^d A^e A^f A^g \nabla_{def} R^a{}_{bgc} - \frac{1}{36}\,A^d A^e A^f A^g R^a{}_{deh} \nabla_h R_{fbgc}\\
&- \frac{1}{45}\,A^d A^e A^f A^g R^a{}_{hdb} \nabla_e R_{fcgh} - \frac{1}{45}\,A^d A^e A^f A^g R_{dbeh} \nabla^a R_{fcgh} - \frac{1}{45}\,A^d A^e A^f A^g R_{dbeh} \nabla_f R^a{}_{hgc} + \frac{1}{60}\,A^d A^e A^f A^g \nabla^a{}_{de} R_{fbgc}\Bigg) + \mathcal{O}\left(\epsilon^6\right)\end{aligned}$$

$$
\begin{aligned}
\frac{d^3x^a}{ds^3} = (-1)\ B^bB^cB^d \Bigg(& \frac{1}{2}\,A^e\nabla_b R^a{}_{ced} + \frac{2}{15}\,A^eA^fR^a{}_{ebg}R_{fcdg} + \frac{8}{15}\,A^eA^fR^a{}_{beg}R_{fcdg} - \frac{2}{15}\,A^eA^fR^a{}_{bcg}R_{edfg} + \frac{3}{10}\,A^eA^f\nabla_{eb}R^a{}_{cfd} + \frac{1}{10}\,A^eA^f\nabla_{bc}R^a{}_{efd} \\
& + \frac{2}{5}\,A^eA^fR^a{}_{geb}R_{fcdg} + \frac{1}{20}\,A^eA^f\nabla^a{}_bR_{ecfd} + \frac{1}{9}\,A^eA^fA^gR_{ebch}\nabla_dR^a{}_{fgh} + \frac{1}{9}\,A^eA^fA^gR_{ebch}\nabla_hR^a{}_{fgd} + \frac{2}{9}\,A^eA^fA^gR_{ebch}\nabla_fR^a{}_{dgh} + \frac{2}{9}\,A^eA^fA^gR_{ebch}\nabla_fR^a{}_{hgd} \\
& + \frac{1}{9}\,A^eA^fA^gR_{ebch}\nabla^aR_{fdgh} + \frac{1}{9}\,A^eA^fA^gR^a{}_{beh}\nabla_cR_{fdgh} + \frac{2}{9}\,A^eA^fA^gR^a{}_{beh}\nabla_fR_{gcdh} - \frac{1}{18}\,A^eA^fA^gR^a{}_{beh}\nabla_hR_{fcgd} + \frac{1}{9}\,A^eA^fA^gR^a{}_{heb}\nabla_cR_{fdgh} \\
& + \frac{2}{9}\,A^eA^fA^gR^a{}_{heb}\nabla_fR_{gcdh} - \frac{1}{18}\,A^eA^fA^gR^a{}_{heb}\nabla_hR_{fcgd}\Bigg) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^4x^a}{ds^4} = (-1)\ B^bB^cB^dB^e \Bigg(& \frac{4}{9}\,A^fR^a{}_{bcg}R_{fdeg} + \frac{1}{9}\,A^fA^gR^a{}_{bch}\nabla_dR_{fegh} + \frac{2}{9}\,A^fA^gR^a{}_{bch}\nabla_fR_{gdeh} - \frac{1}{18}\,A^fA^gR^a{}_{bch}\nabla_hR_{fdge} + \frac{1}{9}\,A^fA^gR_{fbch}\nabla_dR^a{}_{geh} + \frac{5}{9}\,A^fA^gR_{fbch}\nabla_dR^a{}_{hge} \\
& + \frac{2}{9}\,A^fA^gR_{fbch}\nabla_gR^a{}_{deh} + \frac{4}{9}\,A^fA^gR_{fbch}\nabla_hR^a{}_{dge} + \frac{1}{9}\,A^fA^gR_{fbch}\nabla^aR_{gdeh} + \frac{2}{3}\,A^fA^gR_{fbch}\nabla_dR^a{}_{egh} + \frac{1}{3}\,A^fA^gR^a{}_{bfh}\nabla_cR_{gdeh} + \frac{1}{3}\,A^fA^gR^a{}_{hfb}\nabla_cR_{gdeh}\Bigg) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}
$$

$$
\frac{d^5x^a}{ds^5} = (-1)\ B^bB^cB^dB^eB^f\left(\frac{2}{3}\,A^gR^a{}_{bch}\nabla_dR_{gefh} + A^gR_{gbch}\nabla_dR^a{}_{efh}\right) + \mathcal{O}\left(\epsilon^6\right)
$$

$$
\frac{d^6x^a}{ds^6} = 0 + \mathcal{O}\left(\epsilon^6\right)
$$

```
# --- the series solution to the geodesic inital value problem ------------------------------------

sol:=@(deriv00) + s @(deriv01) + (1/2) s**2 @(deriv02) + (1/6) s**3 @(deriv03)
               + (1/24) s**4 @(deriv04) + (1/120) s**5 @(deriv05) + (1/720) s**6 @(deriv06):

@substitute!(sol)(A^{a} -> x^a, B^{a}-> \thedotx^a); "sol.trn"
@print["\Btag{01}x^{a}(s)="~@(sol)~"+\BigO{\eps^6}\Etag{01}"];
```

The Taylor series solution to the geodesic equation

$$0 = \frac{d^2x^a}{ds^2} + \Gamma^a_{bc}(x)\frac{dx^b}{ds}\frac{dx^c}{ds}$$

subject to the initial condtions $x^a(s) = x^a$ and $dx^a/ds = \dot{x}^a$ at $s = 0$, is

$$\begin{aligned}
x^a\left(s\right) = \Bigg(& x^a + s\dot{x}^a - \frac{1}{2}s^2\dot{x}^b\dot{x}^c\Bigg(\frac{2}{3}x^dR^a{}_{bdc} + \frac{1}{6}x^dx^e\nabla_bR^a{}_{dec} + \frac{1}{3}x^dx^e\nabla_dR^a{}_{bec} + \frac{1}{12}x^dx^e\nabla^aR_{dbec} + \frac{8}{45}x^dx^ex^fR^a{}_{deg}R_{fbcg} - \frac{4}{45}x^dx^ex^fR^a{}_{bdg}R_{ecfg} \\
& - \frac{2}{45}x^dx^ex^fR^a{}_{dbg}R_{ecfg} + \frac{1}{10}x^dx^ex^f\nabla_{db}R^a{}_{efc} + \frac{1}{10}x^dx^ex^f\nabla_{de}R^a{}_{bfc} - \frac{2}{45}x^dx^ex^fR^a{}_{gdb}R_{ecfg} + \frac{1}{20}x^dx^ex^f\nabla^a{}_dR_{ebfc} + \frac{4}{45}x^dx^ex^fx^gR_{dbch}\nabla_eR^a{}_{fgh} \\
& + \frac{1}{30}x^dx^ex^fx^gR^a{}_{deh}\nabla_bR_{fcgh} + \frac{4}{45}x^dx^ex^fx^gR^a{}_{deh}\nabla_fR_{gbch} - \frac{2}{45}x^dx^ex^fx^gR^a{}_{bdh}\nabla_eR_{fcgh} - \frac{1}{45}x^dx^ex^fx^gR^a{}_{dbh}\nabla_eR_{fcgh} - \frac{1}{45}x^dx^ex^fx^gR_{dbeh}\nabla_cR^a{}_{fgh} \\
& - \frac{2}{45}x^dx^ex^fx^gR_{dbeh}\nabla_fR^a{}_{cgh} - \frac{1}{45}x^dx^ex^fx^gR_{dbeh}\nabla_fR^a{}_{gch} + \frac{1}{30}x^dx^ex^fx^g\nabla_{deb}R^a{}_{fgc} + \frac{1}{45}x^dx^ex^fx^g\nabla_{def}R^a{}_{bgc} - \frac{1}{36}x^dx^ex^fx^gR^a{}_{deh}\nabla_hR_{fbgc} \\
& - \frac{1}{45}x^dx^ex^fx^gR^a{}_{hdb}\nabla_eR_{fcgh} - \frac{1}{45}x^dx^ex^fx^gR_{dbeh}\nabla^aR_{fcgh} - \frac{1}{45}x^dx^ex^fx^gR_{dbeh}\nabla_fR^a{}_{hgc} + \frac{1}{60}x^dx^ex^fx^g\nabla^a{}_{de}R_{fbgc}\Bigg) - \frac{1}{6}s^3\dot{x}^b\dot{x}^c\dot{x}^d\Bigg(\frac{1}{2}x^e\nabla_bR^a{}_{ced} \\
& + \frac{2}{15}x^ex^fR^a{}_{ebg}R_{fcdg} + \frac{8}{15}x^ex^fR^a{}_{beg}R_{fcdg} - \frac{2}{15}x^ex^fR^a{}_{bcg}R_{edfg} + \frac{3}{10}x^ex^f\nabla_{eb}R^a{}_{cfd} + \frac{1}{10}x^ex^f\nabla_{bc}R^a{}_{efd} + \frac{2}{5}x^ex^fR^a{}_{geb}R_{fcdg} + \frac{1}{20}x^ex^f\nabla^a{}_bR_{ecfd} \\
& + \frac{1}{9}x^ex^fx^gR_{ebch}\nabla_dR^a{}_{fgh} + \frac{1}{9}x^ex^fx^gR_{ebch}\nabla_hR^a{}_{fgd} + \frac{2}{9}x^ex^fx^gR_{ebch}\nabla_fR^a{}_{dgh} + \frac{2}{9}x^ex^fx^gR_{ebch}\nabla_fR^a{}_{hgd} + \frac{1}{9}x^ex^fx^gR_{ebch}\nabla^aR_{fdgh} + \frac{1}{9}x^ex^fx^gR^a{}_{beh}\nabla_cR_{fdgh} \\
& + \frac{2}{9}x^ex^fx^gR^a{}_{beh}\nabla_fR_{gcdh} - \frac{1}{18}x^ex^fx^gR^a{}_{beh}\nabla_hR_{fcgd} + \frac{1}{9}x^ex^fx^gR^a{}_{heb}\nabla_cR_{fdgh} + \frac{2}{9}x^ex^fx^gR^a{}_{heb}\nabla_fR_{gcdh} - \frac{1}{18}x^ex^fx^gR^a{}_{heb}\nabla_hR_{fcgd}\Bigg) \\
& - \frac{1}{24}s^4\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e\Bigg(\frac{4}{9}x^fR^a{}_{bcg}R_{fdeg} + \frac{1}{9}x^fx^gR^a{}_{bch}\nabla_dR_{fegh} + \frac{2}{9}x^fx^gR^a{}_{bch}\nabla_fR_{gdeh} - \frac{1}{18}x^fx^gR^a{}_{bch}\nabla_hR_{fdge} + \frac{1}{9}x^fx^gR_{fbch}\nabla_dR^a{}_{geh} + \frac{5}{9}x^fx^gR_{fbch}\nabla_dR^a{}_{hge} \\
& + \frac{2}{9}x^fx^gR_{fbch}\nabla_gR^a{}_{deh} + \frac{4}{9}x^fx^gR_{fbch}\nabla_hR^a{}_{dge} + \frac{1}{9}x^fx^gR_{fbch}\nabla^aR_{gdeh} + \frac{2}{3}x^fx^gR_{fbch}\nabla_dR^a{}_{egh} + \frac{1}{3}x^fx^gR^a{}_{bfh}\nabla_cR_{gdeh} + \frac{1}{3}x^fx^gR^a{}_{hfb}\nabla_cR_{gdeh}\Bigg) \\
& - \frac{1}{120}s^5\dot{x}^b\dot{x}^c\dot{x}^d\dot{x}^e\dot{x}^f\left(\frac{2}{3}x^gR^a{}_{bch}\nabla_dR_{gefh} + x^gR_{gbch}\nabla_dR^a{}_{efh}\right)\Bigg) + \mathcal{O}\left(\epsilon^6\right)
\end{aligned}$$

```
# ===============================================================================================
#   export genGamma0*
# ===============================================================================================

com:="open geodesic-ivp.lib":
@run(com){"/Users/leo/local/sh/cdbfile"}:

com:="export geodesic-ivp.lib genGamma02.del":
@run(com){"/Users/leo/local/sh/cdbfile"}:

com:="export geodesic-ivp.lib genGamma03.del":
@run(com){"/Users/leo/local/sh/cdbfile"}:

com:="export geodesic-ivp.lib genGamma04.del":
@run(com){"/Users/leo/local/sh/cdbfile"}:

com:="export geodesic-ivp.lib genGamma05.del":
@run(com){"/Users/leo/local/sh/cdbfile"}:

com:="export geodesic-ivp.lib genGamma06.del":
@run(com){"/Users/leo/local/sh/cdbfile"}:

# ===============================================================================================
#   export the solution of the ivp for use by truncate.cdbp
# ===============================================================================================

com:="export geodesic-ivp.lib sol.trn":
@run(com){"/Users/leo/local/sh/cdbfile"}:
```