Mathellaneous

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A Mathemagical Card Trick

1 Card Trick

> Here I have a normal deck of 52 playing cards... take them and have a look for yourself. I would like you to choose five, any five, of your favourite cards and remove them from the deck. Now, being careful not to show me, pass those five cards to my lovely assistant. She will reveal four of them by placing them face up on the table — the $4\spadesuit$, the $Q\clubsuit$, the $5\heartsuit$ and then the $10\diamondsuit$ — but leave the identity of the remaining card known only to you and herself. And the identity of that card is none other than the $9\spadesuit$!

Few people would fail to be amazed by this card trick, especially when performed live. Many spectators will propose that there is some sleight of hand or secret signalling involved, while the more credulous in the audience may even suspect telepathy! However, I assure you that there is no legerdemain, no underhand communication and certainly no mind reading. The cards are all on the table, so to speak, and the only communication seems to be via the order that they are laid down by my lovely assistant. Thus, the question which I now ask is purely a mathematical one: "How does the trick work?"

As with any worthwhile mathematical problem, it pays to spend some time working towards a solution. So I encourage the reader to immediately grab a pen and the nearest scrap of paper and return to this article only after some serious contemplation. The solution appears below but to delay revealing the spoiler, I will briefly recount some history. The card trick originated from the mind of mathematician and magician William Fitch Cheney, Jr (1894–1974), who was awarded the first ever PhD in mathematics from MIT. Fitch's Five Card Trick, as we shall refer to it, first appeared in print in Wallace Lee's *Math Miracles* [2] way back in 1950. It was not until 1986 though, that the trick was revived by yet another mathematician and magician Art Benjamin. Since then, the trick has achieved more widespread fame, appearing on the `rec.puzzles` newsgroup, in recent articles by Michael Kleber [1] and Colm Mulcahy [3], and even in job interviews!

Despite these public appearances, I have come across few people who have had the opportunity to be amazed by Fitch's Five Card Trick. Hopefully this article will help to rectify the situation, at least amongst the mathematicians of this country. For it serves not only as a beautiful problem and interesting party trick, but also as an instructive and entertaining educational tool, as evidenced by the article *Using a Card Trick to Teach Discrete Mathematics* [4]. My hope is that after passing Fitch's Five Card Trick secrets on to you, that it will then be passed on to others, perhaps in the course of a mathematics lecture.

But now we turn our attention to the secret behind Fitch's Five Card Trick. Congratulations to those who have tried the problem and solved it, commiserations to those who have tried to no avail, and those who have not tried the problem should do so without delay. The problem yields to more than one approach, although those of mathematical inclination invariably use the following three main ideas.

- The first observation we make is that the assistant is handed five cards, whereas there are only four possible suits. Thus, a trivial application of the pigeonhole principle yields the fact that at least two cards must be of the same suit. So the assistant can choose one of these to be the hidden card and communicate its suit by revealing the other card first.
- The second observation is that the assistant can use the natural ordering on the cards, from Ace up to King. Ties can be broken by ordering the suits in bridge, or equivalently alphabetical, order — Clubs, Diamonds, Hearts and Spades — so that the Ace of Clubs is the lowest card in the deck and the King of Spades the highest. After revealing the first card, the assistant has no control over what the remaining three cards are, except that there must be a Low, a Medium and a High with respect to the strict ordering described. Thus, rearranging these three cards can communicate one of six things, such as the numbers from 1 to 6. There are many ways to do this, but my preferred method is the following: $1 \leftrightarrow LMH$, $2 \leftrightarrow LHM$, $3 \leftrightarrow MLH$, $4 \leftrightarrow MHL$, $5 \leftrightarrow HLM$, $6 \leftrightarrow HML$.
- Once I see that first card revealed by my assistant, I have immediately pinned down the suit of the hidden card, which leaves only twelve remaining possibilities for its identity. Unfortunately, this is twice the number of possible messages which I can receive from my assistant with only three cards remaining. Resolving this discrepancy is the most subtle of these three main ideas. The critical observation is that a number from 1 to 6 can always be added onto the value of the first card revealed to give the value of the hidden card. For example, if the same suit pair consists of the $3\spadesuit$ and the $9\spadesuit$, then the assistant should play the 3 first and communicate the number 6 with the remaining three cards. But if the same suit pair was the $3\spadesuit$ and the $10\spadesuit$, then the assistant should play the 10 first and communicate the number 6, where the addition is performed modulo 13. The extra factor of two comes from the choice of which of the same suit pair the assistant chooses to reveal and which to hide.

This solution to the problem is far from unique, although the scheme is certainly one of the easiest to implement between magician and assistant. Having said that, I should warn budding magicians not to perform the trick without much practice, some of which can be gained from the following exercise.

What card is being encoded by the following sequences of four cards?

$2\clubsuit, 6\spadesuit, 6\heartsuit, K\clubsuit$	$9\heartsuit, 3\clubsuit, A\diamondsuit, K\heartsuit$
$4\diamondsuit, K\spadesuit, 3\clubsuit, 10\spadesuit$	$3\spadesuit, A\spadesuit, Q\heartsuit, 8\heartsuit$

2 Maths Problems

Maximizing the Deck

Note that the trick involves the assistant receiving a hand consisting of an *unordered* set of five cards and encoding it as a message consisting of an *ordered* set of four cards. The astute reader may have noticed that there is some slack in this encryption scheme for the following two reasons.

- In any hand where all four suits do not appear, the assistant can encode the hand in more than one way. For example, the unordered set of five cards $\{5\diamondsuit, 2\heartsuit, J\heartsuit, 9\spadesuit, Q\spadesuit\}$

can be encoded as either of the following two ordered sets of four cards: $(J\heartsuit, 9\spadesuit, Q\spadesuit, 5\diamondsuit)$ or $(9\spadesuit, 5\diamondsuit, 2\heartsuit, J\heartsuit)$.

- Any message which contains the card that it encodes will never be used. For example, the assistant will never receive a hand which can be encoded by the message $(3\clubsuit, J\heartsuit, 6\clubsuit, Q\diamondsuit)$.

A mathematician's first instinct upon seeing some slack in the solution to a problem is to grab a loose end and try to tighten it. In this particular problem, this can be achieved by increasing the number of cards in the original deck to more than the standard 52. Thus, the natural question to ask ourselves now is the following.

$\boxed{\text{What is the largest size deck for which Fitch's Five Card Trick can be performed?}}$

Suppose that we wish to perform the trick with a deck of N cards, labelled from 1 up to N. Thus, a hand is simply an unordered set of size five while a message is an ordered set of size four, where all elements are integers from 1 to N. If we denote the set of hands by H and the set of messages by M, it is clear that the assistant's role in the trick is merely to apply some prearranged function which takes a hand in H and outputs a message in M. The magician's role in the card trick is to determine the input hand, and hence the identity of the hidden card, of such a function given only the output message. For this to be possible, it is clear that the function must be one-to-one. One consequence of this observation is the fact that there must be at least as many messages as there are hands. This yields the following non-trivial bound.

$$|H| \leq |M| \Rightarrow \frac{N!}{5!(N-5)!} \leq \frac{N!}{(N-4)!} \Rightarrow N \leq 124.$$

Although $N \leq 124$ is a necessary condition for the card trick to work, it may or may not be sufficient. Faced with this upper bound on the number of cards, the obvious path from here is to find a lower bound to match. In other words, we would like to show that the card trick does, in fact, work with a deck containing as many as 124 cards. Given the above discussion, the problem can be presented in the following more visual form.

Is there a way to pair up each of the unordered sets in the left column with a unique ordered set in the right column in such a way that each set from the left column contains its partner set from the right column?

unordered subsets of size 5 with elements from $1, 2, \ldots, 124$	ordered subsets of size 4 with elements from $1, 2, \ldots, 124$
$\{1, 2, 3, 4, 5\}$	$(1, 2, 3, 4)$
$\{1, 2, 3, 4, 6\}$	$(1, 2, 3, 5)$
$\{1, 2, 3, 4, 7\}$	$(1, 2, 3, 6)$
$\vdots$	$\vdots$
$\{120, 121, 122, 123, 124\}$	$(121, 122, 123, 124)$

Indeed, the answer to the above question is in the affirmative, thus proving that Fitch's Five Card Trick can accommodate a deck with 124 cards. The proof of this fact is a beautiful application of Hall's Marriage Theorem, a staple result for combinatorialists. The following statement of the theorem and accompanying sketch proof are for those few whom are yet to be fully acquainted with it, but most readers should be familiar enough with the theorem to proceed beyond without delay.

Hall's Marriage Theorem: Suppose that there are n boys, each of whom wishes to marry a girl that he knows. This is possible if and only if every set of k boys knows at least k girls between them, for all possible k.[a]
Sketch proof: Let us say that a set of boys satisfies the Hall Marriage Condition if every set of k boys knows at least k girls between them, for all possible k. It is clear that if the boys can be married off, then the Hall Marriage Condition must be satisfied, but the reverse implication is the real meat of the theorem. The following is a sketch proof by induction on the number of boys, the base case of one boy being trivial. Now suppose that the theorem is true for $1, 2, \ldots, n$ boys and consider a set of $n + 1$ boys which satisfies the Hall Marriage Condition. One of the following two scenarios must occur.

- Every group of k boys knows at least $k + 1$ girls, for all $k \leq n$.
  Take any boy and marry him to a girl he knows, of which there are at least 2. Then after removing this happily married couple from the picture, the Hall Marriage Condition still holds for the remaining set of n boys. So by the induction hypothesis, the rest of these boys can also be happily married.
- There exists a group of k boys which knows exactly k girls, for $k \leq n$.
  Note that the Hall Marriage Condition is satisfied by this set of k boys, so by the induction hypothesis, we can marry them off. Then after removing these k happily married couples from the picture, the Hall Marriage Condition still holds for the remaining set of $n+1-k$ boys, a fact which the interested reader should verify. So by the induction hypothesis, the rest of these boys can also be happily married.

[a] The majority of expositions on Hall's Marriage Theorem involve a number of girls who wish to find husbands for themselves. Hopefully, the statement provided here will help to restore the balance!

In order to use Hall's Marriage Theorem in this context, it is helpful to think of the ordered sets in the left column of the table as the boys and the unordered sets in the right column as the girls. In particular, we will declare that a boy and girl know each other if and only if the boy set contains the girl set. Of course, all of this information can be neatly represented as a bipartite graph where the 124 vertices in each partition represent the boys and the girls, while edges represent mutual acquaintance. This particular graph has a great deal of structure, not least the fact that every vertex has degree exactly 120, as the reader can easily verify. Suppose now that there is a set of k boys who know between them m girls. Then there must be $120k$ edges emanating from those k boy vertices in the graph and the m girl vertices must on average have degree at least $120k/m$. In the case $m < k$, this value is greater than 120, contradicting the fact that every vertex in the graph has degree exactly 120. So $m \geq k$ as required, proving that the Hall Marriage Condition is satisfied for this particular set of boys and girls. Thus, we conclude that the required function exists.

Constructive Proof

We have proved, somewhat surprisingly, that Fitch's Five Card Trick can be performed using a deck of 124 cards, but no more. But even the purest of mathematicians would be uneasy with this proof for to actually perform the trick requires more than knowledge of the *existence* of a strategy. In fact, not only do we need an explicit example, but for practical reasons, it is convenient to have one which is "easy to implement".

Can you find an "easy to implement" strategy to perform Fitch's Five Card Trick for a deck with 124 cards?

Of course, what we mean by "easy to implement" is subject to debate, but the following scheme is arguably the simplest known. Suppose that the hand drawn consists of the five cards $1 \leq c_1 < c_2 < c_3 < c_4 < c_5 \leq 124$. The lovely assistant should keep the card c_m hidden, where m is chosen to satisfy $m \equiv c_1 + c_2 + c_3 + c_4 + c_5$, where this and subsequent calculations are considered modulo 5. If we let s denote the sum of the four revealed cards, it is clear that $c_m + s \equiv m$, so that $c_m \equiv m - s$. In other words, one of the following five cases must occur:

- c_m is the smallest card in the hand and satisfies $c_m \equiv 1 - s$;
- c_m is the second smallest card in the hand and satisfies $c_m \equiv 2 - s$;
- c_m is the third smallest card in the hand and satisfies $c_m \equiv 3 - s$;
- c_m is the fourth smallest card in the hand and satisfies $c_m \equiv 4 - s$; or
- c_m is the largest card in the hand and satisfies $c_m \equiv 5 - s$.

A far more succinct way to think about the matter is as follows. If the remaining 120 cards, apart from the four revealed, are renumbered from 1 to 120 in increasing order of size, then $c_m \equiv 1 - s$ in the new labelling system. Since there are only $120/5 = 24$ possible labels with the given value modulo 5, the four cards revealed can be ordered in any of $4! = 24$ ways to communicate the new label of the hidden card c_m. It is a simple matter to convert back from the new to the old labels on the cards.

3 Variations on a Theme

Ups and Downs

After a few repeat performances with the same audience, Fitch's Five Card Trick may start to lose some of its shine. This little variation on the original trick due to Colm Mulcahy [3] can add a little more spice to the show. It is based on the fact that numbers can be communicated in binary by placing the cards face up or face down. Just as before, there are three main ideas behind the trick, two of which remain exactly the same as before. The pigeonhole principle guarantees that we can communicate the suit of the hidden card using the first card revealed, while the choice of which card to hide means that we need only communicate a number from 1 through to 6. This is now a simple task in binary, using a face down card to represent 0 and a face up card to represent 1. More explicitly, the correspondence is as follows: $1 \leftrightarrow DDU$, $2 \leftrightarrow DUD$, $3 \leftrightarrow DUU$, $4 \leftrightarrow UDD$, $5 \leftrightarrow UDU$, $6 \leftrightarrow UUD$.

Two notable observations can help to spice up this variation even more. Firstly, since the UUU combination is never used here, the assistant can choose to switch from the original trick to this variation mid-performance. Mulcahy suggests using the line, "Should we make it harder this time, and only show some of the cards?" Secondly, the DDD combination is also never used here, so the first face up card among the three binary digits can always be used to give the suit, thereby making the first card redundant. Thus, only three cards need ever be shown, either face up or face down, while the fourth card can be completely ignored.

What card is being encoded by the following sequences of four cards, using the Ups and Downs variation?

$2\clubsuit, 6\spadesuit$, DOWN, $K\clubsuit$	$9\heartsuit, 3\clubsuit$, DOWN, DOWN
$4\diamondsuit$, DOWN, $3\clubsuit$, $DOWN$	$3\spadesuit, A\spadesuit, Q\heartsuit$, DOWN

Suit Alterations

A shortcoming of the method given above to perform Fitch's Five Card Trick is the fact that the suit of the first card revealed will always match that of the hidden card. It is altogether likely that shrewd audience members will pick up on this feature upon repeat performances. It would be nice to tweak the above scheme so that the card which determines the suit of the hidden card is not always revealed first. This can be achieved using the following scheme. Sum the values of the four cards revealed and reduce modulo 4 to obtain the number 1, 2, 3 or 4. Then use that position to communicate the suit of the hidden card and the remaining three to communicate the value. For example, consider the example from the beginning of this article — $4\spadesuit, Q\clubsuit$, $5\heartsuit$ and $10\diamondsuit$. Summing the values of the cards gives $4+12+5+10 \equiv 3$ modulo 4. Thus, the $5\heartsuit$ determines the suit of the hidden card, the remaining three cards $4\spadesuit, Q\clubsuit, 10\diamondsuit$ are in LHM order, and the hidden card must be the $8\heartsuit$.

What card is being encoded by the following sequences of four cards using the Suit Alterations variation?

$2\clubsuit, 6\spadesuit, 6\heartsuit, K\clubsuit$	$9\heartsuit, 3\clubsuit, A\diamondsuit, K\heartsuit$
$4\diamondsuit, K\spadesuit, 3\clubsuit, 10\spadesuit$	$3\spadesuit, A\spadesuit, Q\heartsuit, 8\heartsuit$

Cards and Coins

In Kleber's article [1], he mentions that Elwyn Berlekamp performs the trick with a deck of size 64, with the additional feat of guessing the result of a coin flipped by an audience member. The trick works in a similar fashion to the scheme proposed for 124 cards above, but upon removing the four cards, only 60 remain rather than 120, and the result of the coin flip is encoded using the remaining bit of information

4 Further Thoughts

Generalizing Fitch's Five Card Trick

Fitch's Five Card Trick originally involved five cards in order to accommodate the standard deck size of 52 cards. However, if we are to consider decks of arbitrary sizes, then there is nothing to stop us from also considering in all generality an n card trick, where the audience member may remove n cards from the deck. We can even generalize the number of cards revealed by the assistant to be some value r, so that the magician must then be able to name the remaining $n-r$ cards. Fortunately, a completely analogous analysis to the one performed above, using Hall's Marriage Theorem, guarantees the existence of strategies for certain values of N, where N is the number of cards in the deck. Let us call this generalized version Fitch's (n, r) Card Trick, so that the $(5, 4)$ case corresponds to the original version.

A strategy exists for Fitch's (n, r) Card Trick if and only if

$$\frac{(N-r)!}{(N-n)!} \geq n!.$$

However, similar to the analysis performed earlier, the proof shows the existence without a concrete construction. Thus, it would be nice to have an answer to the following question.

For arbitrary values of n, r and N satisfying the above inequality, is there an "easy to implement" strategy?

Perfect Card Tricks

One of the pleasing properties about performing Fitch's Five Card Trick with the maximal deck size of 124 is that the number of possible hands is exactly equal to the number of possible messages — there is absolutely no slack in the encryption scheme. More generally, if we were to perform Fitch's $(n, n-1)$ Card Trick, the maximal deck size can be obtained using the bound given above.

$$\frac{(N-n+1)!}{(N-n)!} \geq n! \Rightarrow N \leq n! + n - 1.$$

When equality holds, in other words $N = n! + n - 1$, we are again in the pleasing situation of having an equal number of possible hands and possible messages. Following the parallel with perfect codes from coding theory, let us call Fitch's (n, p) Card Trick a *perfect card trick* if there exists a deck size for which the number of hands is equal to the number of messages. A perfect card trick results from equality in the bound given earlier, so finding them boils down to solving the following Diophantine equation.

For which positive integer values of n and p does there exist a positive integer N such that

$$\frac{(N-p)!}{(N-n)!} = n! \quad ?$$

I have not had the time to solve or research the problem, so I do not know in which part of the spectrum from trivial to notoriously impossible the problem lies.

How Many Card Tricks?

In the articles [1] and [4], the authors raise the combinatorial question of counting the number of strategies that exist for performing Fitch's Five Card Trick. Of course, the problem can be answered for the standard deck of 52 cards or for the maximal deck of 124 cards. It can also be generalized to Fitch's (n, r) Card Trick. It probably makes sense to only count strategies as equivalent if they are related by renumbering the underlying deck of cards.

How many strategies modulo equivalence exist to perform Fitch's (n, r) Card Trick?

More Card Tricks

This particular gem is one of only a small number of card tricks to my knowledge that relate to non-trivial mathematics. I would be very interested to hear of any others from the readership of the Gazette.

References

[1] M. Kleber, *The Best Card Trick*, The Mathematical Intelligencer **24** (2002).
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