Puzzle corner

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Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner. Each issue will include a handful of entertaining puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to N.Do@ms.unimelb.edu.au or send paper entries to: Gazette of the AustMS, Birgit Loch, Department of Mathematics and Computing, University of Southern Queensland, Toowoomba, Qld 4350, Australia.

The deadline for submission of solutions for Puzzle Corner 11 is 1 May 2009. The solutions to Puzzle Corner 11 will appear in Puzzle Corner 13 in the July 2009 issue of the *Gazette*.

Bags and eggs

If you have 20 bags, what is the minimum number of eggs required so that you can have a different number of eggs in each bag?

Photo: Gerla Braklee

Area identity

Suppose that M and N are points on the sides AB and BC of the square $ABCD$ such that $AM = 2MB$ and $BN = 3NC$. Let AN and DM meet at P, AN and CM meet at Q, and CM and DN meet at R. Prove the identity

$$\text{Area}(AMP) + \text{Area}(BMQN) + \text{Area}(CNR) = \text{Area}(DPQR).$$

Hint: Of course, it should be possible to calculate each of the individual areas — but it should be possible to solve this puzzle without doing so!

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Factorial fun

The numbers $1!, 2!, 3!, \ldots, 100!$ are written on a blackboard. Is it possible to erase one of the numbers so that the product of the remaining 99 numbers is a perfect square?

Highway construction

A highway is being built between two cities which are 100 kilometres apart. In the first month, one kilometre of the highway is built. If X kilometres of the highway have been built by the start of a given month, then $\frac{1}{X^{100}}$ more kilometres of highway are built during that month. Will the highway construction ever be finished?

Photo: Herman Brinkman

Busy bee

A bee flies along a path of length four metres, ending precisely where it began. Show that this path is contained in some sphere of radius one metre.

Coin-flipping games

(1) You have a bent coin which lands heads with probability $0 < p < 1$ and tails with probability $1 - p$. Can you devise a coin-flipping game between two players so that each player has probability $\frac{1}{2}$ of winning?
(2) You have a fair coin which lands heads with probability $\frac{1}{2}$ and tails with probability $\frac{1}{2}$. Can you devise a coin-flipping game between two players so that one player has probability $\frac{1}{3}$ of winning?
(3) You have a fair coin which lands heads with probability $\frac{1}{2}$ and tails with probability $\frac{1}{2}$. Can you devise a coin-flipping game between two players so that one player has probability $\frac{1}{\pi}$ of winning?

Solutions to Puzzle Corner 9

The \$50 book voucher for the best submission to Puzzle Corner 9 is awarded to Stephen Howe.

Lucky lottery

Solution by Samuel Mueller: First, note that it is impossible to win \$49, since any two permutations of the numbers from 1 to 50 which agree in 49 places must also agree in the remaining place. So there are at most 50 possible amounts that a player can win, namely \$0, \$1, \$2, ..., \$48, and \$50. So if each of the 50 players wins a different amount of money, then one and only one must win \$50 and the jackpot.

Ultramagic square

Solution by Joachim Hempel: The ten primes between 40 and 81 are 41, 43, 47, 53, 59, 61, 67, 71, 73 and 79. If one of these primes, say p, lies in the ith row and the jth column where $i \neq j$, then the product of the numbers in the ith row is divisible by p, while the product of the numbers in the ith column cannot be divisible by p. It follows that these primes have to lie on the diagonal of the grid. Since there are only nine places available for these ten numbers, there does not exist an ultramagic square.

Note: It is a simple matter to generalise to the notion of $n \times n$ ultramagic squares. It would be interesting to know for which positive integers n there exists an $n \times n$ ultramagic square.

Cakes and boxes

Solution by Alan Jones:

(a) Denote the triangle by ABC, let the angle at A be a and let the angle at B be $3a$. Let D be the point on AC such that $\angle ABD = a$. By construction, the triangle ABD is isosceles with $AD = BD$. Furthermore, we have $\angle BDC = \angle DBC = 2a$, so that the triangle BCD is also isosceles with $BC = DC$. Cutting off the triangle ABD and placing it so that A moves to D and D moves to B achieves the desired result.

(b) Denote the triangle by ABC, let the angle at A be a and the angle at B be $2a$. Let D be the point on AC such that $BC = DC$ and let E be the point on AB such that $\angle AED = a$. By construction, the triangle AED is isosceles with $AD = ED$. Furthermore, we have $\angle BDE = \angle DBE$, so that the triangle BDE is isosceles with $BE = DE$. Cutting off the the triangle AED and placing it so that A moves to E and D moves to B achieves the desired result.

Golden circle

Solution by Ross Atkins: Consider the sequence of points $P_0, P_1, P_2, \ldots$ on a circle whose circumference is equal to the golden ratio ϕ such that P_{n+1} is one unit of arc length along from P_n in the clockwise direction for all n. The irrationality of ϕ guarantees that the points $P_0, P_1, P_2, \ldots$ are distinct and that they form a dense subset of the circle. Define the sequence $f(1) = 1$, $f(2) = 2$, and for $n \geq 3$, let $f(n)$ be the smallest number larger than $f(n-1)$ such that $P_{f(n)}$ lies on the arc between $P_{f(n-2)}$ and $P_{f(n-1)}$ containing P_0. The solution to the problem follows if we can prove that P_0 lies on the minor arc between $P_{f(n-1)}$ and $P_{f(n)}$ for all $n \geq 3$, that $\phi|P_0P_{f(n)}| = |P_0P_{f(n-1)}|$ where distance is measured by arc length, and that $f(1), f(2), \ldots$ is the Fibonacci sequence. This can be verified for small values of n, so to continue by induction, let us assume that the statement is true for some $n \geq 3$.

First, observe that $P_{f(n+1)}$ cannot lie on the minor arc between P_0 and $P_{f(n)}$, because it would imply that $P_{f(n+1)-f(n)}$ lies on the minor arc between $P_{f(n-1)}$

and $P_{f(n)}$, contradicting the minimality of $f(n+1)$. Therefore, P_0 lies on the minor arc between $P_{f(n)}$ and $P_{f(n+1)}$.

Next, we assume for the sake of contradiction that $f(n+1) = f(n) + k$, where $k < f(n-1)$. Then we have the following chain of equalities, where $m = f(n-2) + k < f(n-2) + f(n-1) = f(n)$.

$$\begin{aligned} |P_{f(n-1)}P_{f(n)}| &= |P_{f(n-1)}P_{f(n)+k}| + |P_{f(n)+k}P_{f(n)}| \\ &= |P_0P_{f(n)+k-f(n-1)}| + |P_0P_k| \\ &= |P_0P_m| + |P_0P_k| \end{aligned}$$

By the inductive hypothesis, $P_{f(n)}$ is closer to P_0 than P_m for any $m < f(n)$. It follows that

$$|P_{f(n-1)}P_{f(n)}| = |P_0P_m| + |P_0P_k| > |P_0P_{f(n)}| + |P_0P_{f(n-1)}| = |P_{f(n-1)}P_{f(n)}|,$$

which yields the desired contradiction. Now we observe that $P_{f(n)+f(n-1)}$ must lie on the minor arc between $P_{f(n-1)}$ and $P_{f(n)}$, and we may now conclude that $f(n+1) = f(n) + f(n-1)$.

Finally, we have

$$\begin{aligned} |P_0P_{f(n+1)}| &= |P_0P_{f(n)+f(n-1)}| = |P_0P_{f(n-1)}| - |P_0P_{f(n)}| \\ &= \phi|P_0P_{f(n)}| - |P_0P_{f(n)}| = (\phi-1)|P_0P_{f(n)}| = \frac{1}{\phi}|P_0P_{f(n)}|, \end{aligned}$$

which completes the induction.

Robots in mazes

Solution by Stephen Howe:

(1) We will prove that on an $n \times n$ chessboard, there are more bad mazes than good mazes for $n \geq 2$. First, note that there are $2n^2 - 2n$ possible positions for walls on the interior of the chessboard, so there are 2^{2n^2-2n} mazes. For a maze M, consider the graph G with a vertex corresponding to each square of the chessboard, with two vertices joined by an edge if and only if there is no wall between the corresponding squares. If $e(G)$ is the number of edges in G, then the number of walls in M is $2n^2 - 2n - e(G)$. When M is a good maze, G is connected and so contains at least $n^2 - 1$ edges. Therefore, the number of walls in a good maze is at most $(2n^2 - 2n) - (n^2 - 1) = (n-1)^2$. If we let A_n denote the number of good mazes, then

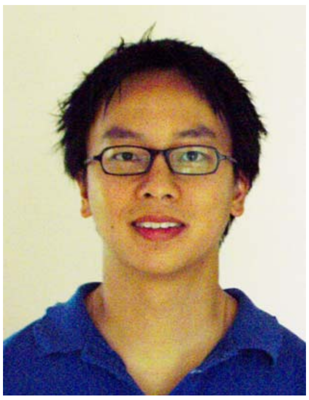
$$A_n \leq \sum_{k=0}^{(n-1)^2} \binom{2n^2-2n}{k}.$$

So, for $n \geq 2$, we have

$$\begin{aligned} 2A_n &\leq 2 \sum_{k=0}^{(n-1)^2} \binom{2n^2-2n}{k} \\ &= \sum_{k=0}^{(n-1)^2} \binom{2n^2-2n}{k} + \sum_{k=0}^{(n-1)^2} \binom{2n^2-2n}{2n^2-2n-k} \\ &< \sum_{k=0}^{2n^2-2n} \binom{2n^2-2n}{k} = 2^{2n^2-2n}. \end{aligned}$$

So the number of good mazes is less than half the number of mazes altogether. Since every maze is either good or bad, there must be more bad mazes than good mazes.

(2) Let $M_1, M_2, \ldots, M_N$ be the list of all proper mazes. If P and Q are two programs, we write the program P followed by Q as PQ. Let P_1 be a program which takes the robot from the start square in M_1 to the finish square in M_1. Next, let P_2 be a program such that P_1P_2 takes the robot from the start square in M_2 to the finish square in M_2. We inductively define P_k to be a program such that $P_1P_2 \ldots P_k$ takes the robot from the start square in M_k to the finish square in M_k. It should be clear that, at every step, it is possible to define the program P_k. Furthermore, if the robot is in the maze M_k, then they will be on the finish square after running the program $P_1P_2 \ldots P_k$. Therefore, the program $P_1P_2 \ldots P_N$ satisfies the conditions of the problem.



Norman is a PhD student in the Department of Mathematics and Statistics at The University of Melbourne. His research is in geometry and topology, with a particular emphasis on the study of moduli spaces of algebraic curves.