Puzzle corner

Norman Do[*]

Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to ndo@math.mcgill.ca or send paper entries to: Gazette of the AustMS, Birgit Loch, Department of Mathematics and Computing, University of Southern Queensland, Toowoomba, Qld 4350, Australia.

The deadline for submission of solutions for Puzzle Corner 15 is 1 January 2010. The solutions to Puzzle Corner 15 will appear in the May 2010 issue of the *Gazette*. This particular issue of the Puzzle Corner happens to be my last. However, puzzle enthusiast Ivan Guo will continue to offer mathematical mindbenders as well as prestigious prizes in future editions of the *Gazette*. I would like to thank those who have read, solved and contributed puzzles over the last three years. I hope you continue to enjoy the Puzzle Corner in future.

Dollars and cents

In the years 1984 to 1988, the only Australian coins in use had denominations of 1 cent, 2 cents, 5 cents, 10 cents, 20 cents, 50 cents and 1 dollar. Prove that if it was possible to pay for A cents with B coins, then it was also possible to pay for B dollars with A coins.

Photo: Emily Dallwitz

Fair game

Albert tosses a fair coin 11 times while Betty tosses a fair coin 10 times. What is the probability that Albert obtained more heads than Betty?

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Evaluation

Let $f(n)$ denote the number of times that the digit 9 occurs in the decimal representation of n. Evaluate the sum

$$2^{f(1)} + 2^{f(2)} + 2^{f(3)} + \cdots + 2^{f(1\,000\,000)}.$$

Congruent cakes

Show that it is possible to bake a triangular cake and cut it into 2009 congruent triangular pieces.


Photo: Bruno Wood

Matrix mayhem

(1) Suppose that you are given an $m \times n$ matrix of real numbers and that you are allowed to reverse the signs of all the numbers in any row or column. Prove that you can continue to do this until the sums of the numbers in each row and column are all non-negative.

(2) In an $N \times N$ matrix of numbers, the rows are pairwise distinct, where we consider two rows to be distinct if they differ in at least one entry. Prove that there is a column which can be deleted in such a way that the rows of the resulting matrix are still pairwise distinct.

(3) Let $a_1, a_2, \ldots, a_{100}, b_1, b_2, \ldots, b_{100}$ be distinct real numbers. A 100×100 matrix M is given whose (i, j) entry is $a_i + b_j$. If the product of the numbers in each column is equal to 1, prove that the product of the numbers in each row is equal to -1.

Solutions to Puzzle Corner 13

The \$50 book voucher for the best submission to Puzzle Corner 13 is awarded to Reiner Pope.

Digital deduction

Solution by: Ivan Guo

We begin with the observation that the number of digits in n is equal to $\lfloor \log_{10} n \rfloor + 1$. So the answer to this problem is simply

$$\lfloor \log_{10} 2^{2009} \rfloor + \lfloor \log_{10} 5^{2009} \rfloor + 2.$$

However, since $\log_{10} 2^{2009} + \log_{10} 5^{2009} = \log_{10} 10^{2009} = 2009$ and neither $\log_{10} 2^{2009}$ nor $\log_{10} 5^{2009}$ can be integers, we know that $\lfloor \log_{10} 2^{2009} \rfloor + \lfloor \log_{10} 5^{2009} \rfloor = 2008$. Therefore, the answer to the problem is 2010.

Square, triangle and circle

Solution by: Alan Jones

Construct the equilateral triangle CDY where Y lies inside the square $ABCD$. Then $XADY$ is a parallelogram, so we have the equal lengths $XY = AD = 1$. Therefore Y is the centre of a circle of radius 1 which passes through C, D and X.

Piles of stones

Solution by: Reiner Pope

We will prove that when we begin with n stones, the sum of the numbers will always be $\frac{n(n-1)}{2}$. We will prove this by strong induction, noting that the case of one stone is clearly true. Now suppose that the statement is true for $1, 2, \ldots, n-1$ stones and that we begin with n stones. If the first move consists of dividing the stones into two piles of sizes j and $n-j$, then the final sum of the numbers will be the product $j(n-j)$ written on the board, plus the sum of the numbers for the two smaller games played on the piles of sizes j and $n-j$. So by the induction hypothesis, the final sum of the numbers will be

$$j(n-j) + \frac{j(j-1)}{2} + \frac{(n-j)(n-j-1)}{2} = \frac{n(n-1)}{2}.$$

This shows that when we begin with n stones, the sum of the numbers will always be $\frac{n(n-1)}{2}$. Hence, when we begin with 25 stones, the sum of the numbers will always be 300.

Do you know my number now?

Solution by: Kevin McAvaney

Suppose genius G has the integer 1 and genius H has the integer 2. If H asks first, then G must answer YES. If G asks first, then H must answer NO and before asking G who must then answer YES.

Suppose genius G has the integer 2 and genius H has the integer 3. If G asks first, then H must answer NO. Then G knows that H does not have the integer 1 and must follow with the answer YES. If H asks first, then G must answer NO and H must follow with the answer NO. Then G knows that H does not have the integer 1 and must answer YES.

Assume that if the two integers are k and $k+1$, then the genius with the integer k answers YES after at most $k-1$ NOs when the genius with the even number asks first, and answers YES after at most k NOs when the genius with the odd number asks first.

Now suppose genius G has the integer $k+1$ and genius H has the integer $k+2$.

- If $k+1$ is odd and H asks first, then after k NOs G will know that H does not have the integer k and must answer YES.

- If $k+1$ is even and G asks first, then after k NOs G will know that H does not have the integer k and must answer YES.
- If $k+1$ is odd and G asks first, then after $k+1$ NOs G will know that H does not have the integer k and must answer YES.
- If $k+1$ is even and H asks first, then after $k+1$ NOs G will know that H does not have the integer k and must answer YES.

By induction, if the integers are n and $n+1$, then one of the geniuses will answer affirmatively after at most n negative answers.

Fun with floors

Solution by: Alexander Hanysz

Define an $n \times n$ matrix $A = (a_{ij})$ where $a_{ij} = 1$ if $i^j \leq n$ and $a_{ij} = 0$ otherwise. The first row of A contains the number 1 exactly n times and for $i \geq 2$, the number of times that 1 occurs in the ith row is equal to the largest j such that $i^j \leq n$, that is, $\lfloor \log_i n \rfloor$. Therefore, the sum of the entries in the matrix A is $n + \lfloor \log_2 n \rfloor + \lfloor \log_3 n \rfloor + \cdots + \lfloor \log_n n \rfloor$.

The first column of A contains the number 1 exactly n times and for $j \geq 2$, the number of times that 1 occurs in the jth column is equal to the largest i such that $i^j \leq n$, that is, $\lfloor \sqrt[j]{n} \rfloor$. Therefore, the sum of the entries in the matrix A is also $n + \lfloor \sqrt{n} \rfloor + \lfloor \sqrt[3]{n} \rfloor + \cdots + \lfloor \sqrt[n]{n} \rfloor$ and the result follows by equating the two sums.

Noodling around

Solution by: James East

Write $P(n,k)$ for the probability that, starting with n noodles, we end up with k loops.

(1) At each step, if there are m unlooped noodles remaining — including long noodles that have been created along the way — then joining two ends together will either create a loop or a long noodle, with probabilities $\frac{1}{2m-1}$ and $\frac{2m-2}{2m-1}$ respectively. So we see that

$$P(n,1) = \frac{(2n-2)!!}{(2n-1)!!} \quad \text{and} \quad P(n,n) = \frac{1}{(2n-1)!!}.$$

Here we have used the double factorial notation $m!! = m(m-2)(m-4)\cdots$, where the product ends in 1 or 2 depending on the parity of m. When $n = 100$ we obtain

$$P(100,1) \approx 0.0887 \quad \text{and} \quad P(100,100) \approx 1.500 \times 10^{-187}.$$

(2) Suppose now that $1 < k < n$, and that we end up with k loops. If the first join creates a loop, we must then form $k-1$ loops from the remaining $n-1$ noodles. Otherwise, we must then create k loops from the remaining $n-1$ noodles. Adding the relevant probabilities gives

$$P(n,k) = \frac{1}{2n-1} P(n-1,k-1) + \frac{2n-2}{2n-1} P(n-1,k).$$

Writing $S(n,k) = \frac{(2n-1)!!}{2^{n-k}} P(n,k)$, this recurrence takes the form

$$S(n,k) = S(n-1,k-1) + (n-1)S(n-1,k),$$

where $S(n,1) = (n-1)!$ and $S(n,n) = 1$. In other words, $S(n,k)$ are the Stirling numbers of the first kind. Rearranging gives the following formula for $P(n,k)$.

$$P(n,k) = \frac{2^{n-k} S(n,k)}{(2n-1)!!}.$$

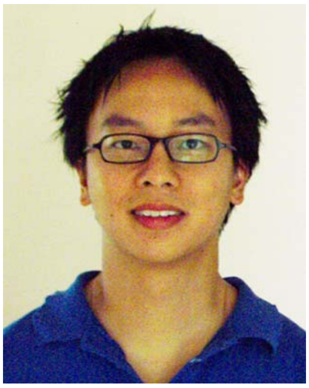
(3) For a fixed n, the expected value for the number of loops is simply

$$\mathrm{E}(n) = \sum_{k=1}^{n} k \cdot P(n,k) = \frac{1}{(2n-1)!!} \sum_{k=1}^{n} k 2^{n-k} S(n,k).$$

With some algebraic trickery, one can show that $\mathrm{E}(n+1) = (2n+1)\mathrm{E}(n)$. Along with the fact that $\mathrm{E}(1) = 1$, this yields the rather satisfying expression

$$\mathrm{E}(n) = \sum_{k=1}^{n} \frac{1}{2k-1}.$$

Putting $n = 100$ we obtain $\mathrm{E}(100) = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{199} \approx 3.2843$.



Norman Do is currently a CRM-ISM Postdoctoral Fellow at McGill University in Montreal. He is an avid solver, collector and distributor of mathematical puzzles. When not playing with puzzles, Norman performs research in geometry and topology, with a particular focus on moduli spaces of curves.