Testing the Effects of Exogenous Determinants on Inefficiency in Panel Stochastic Frontier Models

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11-12 August 2025
MPSS Workshop

Introduction and motivation

- A panel stochastic frontier model is often expressed as

$$y_{it} = \boldsymbol{x}_{it}'\boldsymbol{\beta} - u_{it} + v_{it},$$

  where y_{it} is the output for firm $i = 1, \ldots, N$ at time $t = 1, \ldots, T$, and $u_{it} \geq 0$ represent inefficiencies.

- In the literature, a distributional assumption of u_{it} is often required, and a typical assumption is $\mathcal{N}^+(0, \sigma_u^2)$.

- Information conveyed by a sample of $(\boldsymbol{y}_{it}, \boldsymbol{x}_{it})$ is not enough for estimating u_{it} and its distribution.

- In some applied studies, it is sometimes assumed that $u_{it} \sim \mathcal{N}^+(0, \sigma_u^2)$ where the variance is driven by some determinants via a form such as $\log \sigma_u^2 = \boldsymbol{z}_{it}'\alpha$ (Lai and Kumbhakar, 2018, Econ Letters).

Introduction and motivation

- There is a large body of literature studying the impact of determinant or environmental variables on inefficiencies: Kim and Schmidt (2008, JoE), Amsler, Prokhorov, and Schmidt (2017, JoE), Kutlu, Tran and Tsionas (2019, JoE), Lai and Kumbhakar (2018, EJOR), and Centorrino and Pérez-Urdiales (2023, JoE).

- It is unknown whether $\boldsymbol{z}_{it}$ is significant or not, and whether each variable of $\boldsymbol{z}_{it}$ is significant or not.

- There is no way to justify the analytical form of

$$u_{it} = g(\boldsymbol{z}_{it})$$

- Our goal is to test the significance of $\boldsymbol{z}_{it}$ and the significance of each individual variable of $\boldsymbol{z}_{it}$.

- How to handle nuisance parameters and boundary constraints?

Estimation of the alternative model

- We assume the transient inefficiency is an unknown function of $\boldsymbol{z}_{it}$:

$$
\begin{aligned}
y_{it} &= \boldsymbol{x}_{it}^{\top}\boldsymbol{\beta} - u_{it} + v_{it}, \\
u_{it} &= \delta_i + g(\boldsymbol{z}_{it})
\end{aligned}
$$

  where $\delta_i \sim \mathcal{N}^+(0, \sigma_u^2)$ and $v_{it} \sim \mathcal{N}(0, \sigma_v^2)$.

- $g(\cdot)$ is deterministic, continuous, and nonlinear

- A key challenge is to ensure that $g(\boldsymbol{z}_{it})$ is nonnegative, reflecting the nature of inefficiency.

- We impose an additive structure on $g(\boldsymbol{z}_{it})$ by modeling it as the sum of q separate univariate functions:

$$
g(\boldsymbol{z}_{it}) = \sum_{\ell=1}^{q} g_\ell(z_{it,\ell})
$$

Approximation by Bernstein polynomials

- Each function $g_\ell(z_{it,\ell})$ is approximated as

$$g_\ell(z_{it,\ell}) \approx f(z_{it,\ell}; k)'\boldsymbol{\alpha}_\ell$$

where $f(z_{it,\ell}; k)$ is the $(k+1)$-dimensional vector of Bernstein basis functions

$$f(z_{it,\ell}; k) = \big(b_0(z_{it,\ell}, k), \ldots, b_k(z_{it,\ell}, k)\big)',$$

- Each basis function $b_j(z, k)$ is defined as

$$b_j(z,k) = \binom{k}{j} z^j (1-z)^{k-j}, \quad \text{for } j = 0, \ldots, k.$$

- Basis functions form a partition of unity over the unit interval $[0,1]$, satisfying $\sum_{j=0}^{k} b_j(z,k) = 1$ for all $z \in [0,1]$.

Approximation by Bernstein polynomials

- To ensure compatibility with the Bernstein basis, each determinant $z_{it,\ell}$ must be rescaled to lie within $[0, 1]$.
- Following Wang and Ghosh (2012), we apply an inflated min-max transformation:

$$z^*_{it,\ell} = \frac{z_{it,\ell} - (a_\ell - s_\ell)}{(b_\ell + s_\ell) - (a_\ell - s_\ell)},$$

  where $a_\ell = \min_{i,t} z_{it,\ell}$, $b_\ell = \max_{i,t} z_{it,\ell}$, and s_ℓ is the sample standard deviation of $z_{it,\ell}$ across all i and t.
- The function $g_\ell(z)$ defined on $(0, 1)$ is approximated by

$$B_k(g_\ell; z) = \sum_{j=0}^{k} g_\ell(j/k) \binom{k}{j} z^j (1 - z)^{k-j}.$$

  The target function is a weighted sum of fixed basis functions evaluated at equal distance nodes.

Parameterisation

- The stochastic frontier model becomes

$$y_{it} = \boldsymbol{x}_{it}'\boldsymbol{\beta} - \sum_{\ell=1}^{q} f(z_{it,\ell}; k)'\boldsymbol{\alpha}_\ell - \delta_i + v_{it},$$

- The term $\sum_{\ell=1}^{q} f(z_{it,\ell}; k)'\boldsymbol{\alpha}_\ell$ captures the additive inefficiency effects from all determinants $z_{it,\ell}, \ldots, z_{it,q}$, each represented via a Bernstein polynomial basis.

- The term $\sum_{\ell=1}^{q} f(z_{it,\ell}; k)'\boldsymbol{\alpha}_\ell$ captures the cumulative inefficiency effects from all determinants $z_{it,1}, \ldots, z_{it,q}$, each represented via a Bernstein polynomial basis.

- Let $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1', \boldsymbol{\alpha}_2', \ldots, \boldsymbol{\alpha}_q')'$ denote the full coefficient vector obtained by stacking the Bernstein coefficients across all determinants.

- Restrictions are $\boldsymbol{\alpha} \geq 0$.

Estimation

- The distribution of the composed error term $e_{it} = v_{it} - u_{it}$ depends on the specified distributions for v_{it} and u_{it}.

$$y_{it} = \boldsymbol{x}_{it}'\boldsymbol{\beta} - \sum_{\ell=1}^{q} f(z_{it,\ell}; k)'\boldsymbol{\alpha}_\ell - \delta_i + v_{it},$$

- As $g(\boldsymbol{z}_{it})$ is a deterministic function, the stochastic part of the inefficiency component is given by δ_i.
- The density $f_\varepsilon(\cdot)$ is given by the convolution of the density of v_{it} with the reflected density of δ_i:

$$f_\varepsilon(\varepsilon) = \int_0^\infty f_v(\varepsilon + \delta)\, f_\delta(\delta)\; d\delta.$$

- The likelihood is

$$\mathcal{L}(\boldsymbol{\beta}, \boldsymbol{\alpha}, \boldsymbol{\theta}_u, \boldsymbol{\theta}_v) = \sum_{t=1}^{T} \sum_{i=1}^{N} \log f_\varepsilon \left(y_{it} - \boldsymbol{x}_{it}'\boldsymbol{\beta} + \sum_{\ell=1}^{q} f(z_{it,\ell}; k)'\boldsymbol{\alpha}_\ell \right).$$

Likelihood

- The parameters $(\boldsymbol{\beta}, \boldsymbol{\alpha}, \boldsymbol{\theta}_u, \boldsymbol{\theta}_v)$ are estimated by maximizing this log-likelihood, subject to the constraint $\boldsymbol{\alpha} \geq 0$.
- To choose the optimal order k, we estimate the model over a grid of candidate values and select the one that minimises BIC, computed as

$$\begin{aligned}\text{BIC} = &-2\,\mathcal{L}\big(\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\theta}}_u, \hat{\boldsymbol{\theta}}_v\big)\\ &+ \big(p + q(k+1) + \dim(\boldsymbol{\theta}_u) + \dim(\boldsymbol{\theta}_v)\big)\log(NT),\end{aligned}$$

  where $\mathcal{L}$ denotes the maximized log-likelihood, p is the number of input variables excluding the intercept, and the last term accounts for the total number of model parameters.

Testing the effects of inefficiency determinants

- Rewrite the model in compact vector-matrix form:

$$\mathbf{y} = \mathbf{X}\beta - \mathbf{F}^k\boldsymbol{\alpha} - \mathbf{D}\boldsymbol{\delta} + \mathbf{v}, \tag{1}$$

- Test joint significance

$$H_0 : \boldsymbol{\alpha} = \mathbf{0}, \tag{2}$$

against the alternative that at least one component of $\boldsymbol{\alpha}$ is strictly positive.

- The test is based on the classical F statistic:

$$F = \frac{(\mathrm{SSR}_0 - \mathrm{SSR}_1)/r}{\mathrm{SSR}_1/(NT - p - 1 - r)}, \tag{3}$$

where $\mathrm{SSR}_0 = \hat{\varepsilon}_0'\hat{\varepsilon}_0$ is the sum of squared residuals under H_0.

Estimation

- These residuals are obtained by projecting $\mathbf{y}$ onto the orthogonal complement of the column space of $\mathbf{X}$:

$$\hat{\varepsilon}_0 = \mathbf{M}_\mathbf{X}\mathbf{y}, \text{ and } \mathbf{M}_\mathbf{X} = \mathbf{I}_{NT} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$$

- The number of restrictions in the F stats is $r = q(k+1)$, corresponding to the Bernstein coefficients that are set to zero under H_0.
- The unrestricted sum of squared residuals appearing in the F stats is $\text{SSR}_1 = \hat{\varepsilon}_1'\hat{\varepsilon}_1$, where $\hat{\varepsilon}_1 = \mathbf{M}_\mathbf{X}\mathbf{y} + \mathbf{M}_\mathbf{X}\mathbf{F}^k\hat{\alpha}$.
- $\hat{\alpha}$ is obtained by solving the following constrained least squares problem:

$$\min_{\boldsymbol{\alpha}\geq 0}\left\{\left(\mathbf{M}_\mathbf{X}\mathbf{y} + \mathbf{M}_\mathbf{X}\mathbf{F}^k\alpha\right)'\left(\mathbf{M}_\mathbf{X}\mathbf{y} + \mathbf{M}_\mathbf{X}\mathbf{F}^k\alpha\right)\right\}.$$

Nuisance parameters

- The parameters contained in the distributions of δ_i and v_{it} (denoted as $\boldsymbol{\theta}_u$ and $\boldsymbol{\theta}_v$) are nuisance parameters.

- Although the null distribution of the F-stats depends on $\boldsymbol{\theta}_u$ and $\boldsymbol{\theta}_v$, they are not of direct interest in the hypothesis being tested.

- We define the point null hypothesis

$$H_0(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*) : \boldsymbol{\alpha} = \mathbf{0},\ \boldsymbol{\theta}_u = \boldsymbol{\theta}_u^*,\ \boldsymbol{\theta}_v = \boldsymbol{\theta}_v^*, \tag{4}$$

- All inefficiency effects are absent and that the error term parameters (nuisance) take fixed values $\boldsymbol{\theta}_u^*$ and $\boldsymbol{\theta}_v^*$.

- The reference distribution for the F-stats under this point null can be readily obtained using simulation.

Nuisance parameters

- When nuisance parameters are not assumed to be known, the original null hypothesis (H_0) can be expressed as

$$H_0 : \bigcup_{(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*)} H_0(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*),$$

- It does not fix the nuisance parameters at specific values but instead encompasses all possible combinations.
- Accordingly, H_0 can be rejected only if it is rejected at every admissible value of $(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*)$.
- We therefore aim to construct an exact simulation-based test of the joint point null hypothesis.
- We will develop a maximized version that accounts for the nuisance parameters by computing the largest simulated p-value over a grid of plausible values.

Some properties of the test procedure

Proposition 1: Under the point null hypothesis $H_0(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*)$, the F stats defined as

$$F = \frac{(\mathrm{SSR}_0 - \mathrm{SSR}_1)/r}{\mathrm{SSR}_1/(NT - p - 1 - r)},$$

has the same distribution as the random variable

$$\widetilde{F}(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*) = \frac{(\widetilde{\mathrm{SSR}}_0 - \widetilde{\mathrm{SSR}}_1)/r}{\widetilde{\mathrm{SSR}}_1/(NT - p - 1 - r)},$$

where each quantity depends on $(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*)$ through the simulated residual vectors

$$\tilde{\varepsilon}_0 = \mathbf{M_X}(-\mathbf{D}\tilde{\boldsymbol{\delta}} + \tilde{\mathbf{v}}), \quad \tilde{\varepsilon}_1 = \tilde{\varepsilon}_0 + \mathbf{M_X}\mathbf{F}^k\tilde{\boldsymbol{\alpha}}.$$

Remark 1: The minimisation defining SSR_1 and $\widetilde{\mathrm{SSR}}_1$ is purely deterministic. Therefore, both F and $\widetilde{F}(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*)$ are deterministic functions of their corresponding residual vectors

Explanation

- This result implies that even if the null distribution of the F stats is non-standard, it can be simulated exactly for any given values of $\mathbf{X}$, $\mathbf{F}^k$ and nuisance parameters $(\boldsymbol{\theta}_u, \boldsymbol{\theta}_v)$.
- The entire distribution of F under $H_0(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*)$ can be generated through repeated simulation of $\tilde{\varepsilon}_0$ and re-estimation of the constrained model.
- In order to obtain exact p-values without relying on a large number of simulations, we use the Monte Carlo (MC) test framework of Dwass (1957), Barnard (1963) and Birnbaum (1974).

Proposition 2: Assume that the data $\mathbf{y}$ follow the stochastic frontier model in (1), and consider the F stats defined in (3). Let $\widetilde{F}_1(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*), \ldots, \widetilde{F}_{B-1}(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*)$ denote $B-1$ draws from the distribution of $\widetilde{F}(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*)$ under $H_0(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*)$. The Monte Carlo p-value is defined as

$$\tilde{p}(F \,|\, \boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*) = \frac{B - \text{Rank}(F \,|\, \boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*) + 1}{B},$$

where $\text{Rank}(F \,|\, \boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*)$ is given by

$$\text{Rank}(F \,|\, \boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*) = 1 + \sum_{b=1}^{B-1} \mathbf{1}\{F > \widetilde{F}_b(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*)\},$$

- If αB is an integer, then under $H_0(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*)$, the test that rejects H_0 when $\tilde{p}(F \,|\, \boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*) \leq \alpha$ has exact size α; that is,

$$\Pr\left(\tilde{p}(F \,|\, \boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*) \leq \alpha \,|\, H_0(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*)\right) = \alpha.$$

Maximised Monte Carlo Test:

- This proposition provides an exact test under fixed nuisance parameters, while in practice these parameters are typically unknown.
- To address this issue, we adopt the maximised MC (MMC) testing framework of Dufour (2006), which constructs a test that controls size uniformly over a set of plausible values for the nuisance parameters.

Proposition 3:
Let $\tilde{p}(F)$ denote the maximised Monte Carlo (MMC) p-value:

$$\tilde{p}(F) = \sup_{(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*) \in \mathcal{S}} \tilde{p}\big(F \,|\, \boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*\big),$$

where $\tilde{p}\big(F \,|\, \boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*\big)$ is the MC p-value from Proposition 2, and $\mathcal{S}$ is a set of plausible values for the parameter vectors $(\boldsymbol{\theta}_u, \boldsymbol{\theta}_v)$. If αB is an integer, then the test that rejects H_0 whenever $\tilde{p}(F) \leq \alpha$ has level at most α, that is

$$\Pr\big(\tilde{p}(F) \leq \alpha \,|\, H_0\big) \leq \alpha.$$

- To implement the MMC test, we define the search domain $\mathcal{S}$ as a linear path connecting the MLEs of the nuisance parameters obtained under the alternative model and those estimated under the joint null hypothesis.

Testing marginal significance

- To test whether a specific inefficiency determinant $z_{it,\ell}$ has an effect, we consider a null hypothesis that restricts only the corresponding coefficients, while leaving the others unrestricted:

$$H_0^\ell : \boldsymbol{\alpha}_\ell = \mathbf{0}, \tag{5}$$

  for $\ell \in \{1, \ldots, q\}$, against the alternative that at least one element of $\boldsymbol{\alpha}_\ell$ is strictly positive.
- The associated restricted sum of squared residuals is obtained by solving the constrained least squares problem

$$\mathrm{SSR}_{0,\ell} = \min_{\substack{\boldsymbol{\alpha}\geq 0 \\ \boldsymbol{\alpha}_\ell = 0}} \left(\mathbf{M_X y} + \mathbf{M_X F}^k \boldsymbol{\alpha}\right)' \left(\mathbf{M_X y} + \mathbf{M_X F}^k \boldsymbol{\alpha}\right), \tag{6}$$

Test statistic

- The corresponding F stats is

$$F_\ell = \frac{(\mathrm{SSR}_{0,\ell} - \mathrm{SSR}_1)/(k+1)}{\mathrm{SSR}_1/(NT-p-1-r)}, \tag{7}$$

  where SSR_1 is the unrestricted residual sum of squares, computed as in the joint test in Proposition 1
- The null distribution of F_ℓ also depends on $\boldsymbol{\alpha}_{-\ell}$
- We address this additional nuisance parameter problem by introducing a point null hypothesis of the form:

$$H_0^\ell(\boldsymbol{\alpha}^*_{-\ell}) : \boldsymbol{\alpha}_\ell = \mathbf{0},\ \boldsymbol{\alpha}_{-\ell} = \boldsymbol{\alpha}^*_{-\ell},$$

  where $\boldsymbol{\alpha}^*_{-\ell}$ denotes the Bernstein coefficients for all other determinants.
- These values are chosen to satisfy the broader null H_0^ℓ, so that $H_0^\ell(\boldsymbol{\alpha}^*_{-\ell}) \subseteq H_0^\ell$

Test statistic for marginal significance

- We introduce a test statistic in the form of an F-ratio tailored to this point null hypothesis:

$$F_\ell^* = \frac{(\mathrm{SSR}_{0,\ell}^* - \mathrm{SSR}_1)/(k+1)}{\mathrm{SSR}_1/(NT-p-1-r)}, \tag{8}$$

- SSR_1 is the unrestricted residual sum of squares. $\mathrm{SSR}_{0,\ell}^*$ denotes the residual sum of squares under $H_0^\ell(\boldsymbol{\alpha}_{-\ell}^*)$:

$$\mathrm{SSR}_{0,\ell}^* = \left(\mathbf{M_X y} + \mathbf{M_X F}_{-\ell}^k \boldsymbol{\alpha}_{-\ell}^*\right)' \left(\mathbf{M_X y} + \mathbf{M_X F}_{-\ell}^k \boldsymbol{\alpha}_{-\ell}^*\right),$$

- The restriction $\boldsymbol{\alpha}_\ell = \mathbf{0}$ is imposed directly, and the remaining coefficients $\boldsymbol{\alpha}_{-\ell}$ are fixed at the specified values $\boldsymbol{\alpha}_{-\ell}^*$.

Test statistic for marginal significance

- As $H_0^\ell(\boldsymbol{\alpha}_{-\ell}^*)$ imposes stronger restrictions than H_0^ℓ, we have $\mathrm{SSR}_1 \le \mathrm{SSR}_{0,\ell} \le \mathrm{SSR}_{0,\ell}^*$, which in turn implies $F_\ell \le F_\ell^*$. Moreover, since $H_0^\ell(\boldsymbol{\alpha}_{-\ell}^*) \Rightarrow H_0^\ell$, it follows that

$$\Pr\left(F_\ell \ge c \,|\, H_0^\ell\right) \le \Pr\left(F_\ell^* \ge c \,|\, H_0^\ell(\boldsymbol{\alpha}_{-\ell}^*)\right),$$

  for any $c \in \mathbb{R}$.
- This inequality ensures that using the distribution of F_ℓ^* as a reference yields a conservative test for H_0^ℓ.
- With this bounding approach, we actually use $H_0^\ell(\boldsymbol{\alpha}_{-\ell}^*)$ to test H_0^ℓ.
- For further discussion and examples of such bounding procedures, see Dufour and Khalaf (2002) and Luger (2025).

Point null hypothesis

- We now incorporate the error term parameters and consider a fully specified point null hypothesis:

$$H_0^{\ell}(\boldsymbol{\alpha}^*_{-\ell}, \boldsymbol{\theta}^*_u, \boldsymbol{\theta}^*_v) : \boldsymbol{\alpha}_\ell = \mathbf{0},\ \boldsymbol{\alpha}_{-\ell} = \boldsymbol{\alpha}^*_{-\ell},\ \boldsymbol{\theta}_u = \boldsymbol{\theta}^*_u,\ \boldsymbol{\theta}_v = \boldsymbol{\theta}^*_v.$$

- In this formulation, the other Bernstein coefficients and error term parameters are all nuisance and fixed.
- $H_0^{\ell}(\boldsymbol{\alpha}^*_{-\ell})$ can be viewed as the union over the admissible values of the error term parameters parameters:

$$H_0^{\ell}(\boldsymbol{\alpha}^*_{-\ell}) = \bigcup_{\boldsymbol{\theta}^*_u, \boldsymbol{\theta}^*_v} H_0^{\ell}(\boldsymbol{\alpha}^*_{-\ell}, \boldsymbol{\theta}^*_u, \boldsymbol{\theta}^*_v).$$

- We define the simulated test statistic

$$\widetilde{F}^*_\ell(\boldsymbol{\theta}^*_u, \boldsymbol{\theta}^*_v) = \frac{(\widetilde{\mathrm{SSR}}^*_{0,\ell} - \widetilde{\mathrm{SSR}}_1)/(k+1)}{\widetilde{\mathrm{SSR}}_1/(NT-p-1-r)},$$

Proposition 4

Let $\tilde{p}(F_\ell)$ denote the MMC p-value for testing H_0^ℓ, defined as

$$\tilde{p}(F_\ell) = \sup_{(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*) \in \mathcal{S}} \tilde{p}(F_\ell \,|\, \boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*),$$

where $\mathcal{S}$ is a set of plausible values for the error term parameters, and $\tilde{p}(F_\ell \,|\, \boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*)$ is the MC p-value given by

$$\tilde{p}(F_\ell \,|\, \boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*) = \frac{B - \text{Rank}(F_\ell \,|\, \boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*) + 1}{B},$$

with rank $\text{Rank}(F_\ell \,|\, \boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*) = 1 + \sum_{b=1}^{B-1} \mathbf{1}\{F_\ell > \widetilde{F}_{\ell,b}^*(\boldsymbol{\theta}_u^*, \boldsymbol{\theta}_v^*)\}$. If αB is an integer, then the test that rejects H_0^ℓ when $\tilde{p}(F_\ell) \leq \alpha$ has level at most α, that is

$$\Pr\left(\tilde{p}(F_\ell) \leq \alpha \,|\, H_0^\ell\right) \leq \alpha.$$

Test implementation

- To implement the point null hypothesis $H_0^{\ell}(\boldsymbol{\alpha}^*_{-\ell})$, we must specify the fixed value $\boldsymbol{\alpha}^*_{-\ell}$ for the Bernstein coefficients associated with all other inefficiency determinants.
- In our implementation, we set $\boldsymbol{\alpha}^*_{-\ell}$ equal to the vector of estimated coefficients $\hat{\boldsymbol{\alpha}}_{-\ell}$ obtained by solving the constrained least squares problem in (6), where the restriction $\boldsymbol{\alpha}_{\ell} = \mathbf{0}$ is imposed.
- As in the joint test, we restrict the MMC search over error term parameters to a one-dimensional path connecting the MLEs from the full model and the complete null model, where all $\boldsymbol{\alpha}_{\ell}$ are set to zero.

China's Energy Consumption Frontier and Inefficiency

- Sample data: $N = 30$ provinces; and $T = 19$ (2003–2021)
- We model the minimum energy consumption required to produce energy services using a stochastic frontier model:

$$y_{it} = x_{it}'\beta + u_{it} + v_{it},$$

where

$$u_{it} = \delta_i + g(\boldsymbol{z}_{it}).$$

- Frontier represents the minimum level of energy consumption that could be achieved for a given level of output and input mix, assuming full efficiency.
- It is determined by a set of variables:
  Energy price, GDP, average household size, population, climate (proxied by heating and cooling degree days), transport infrastructure (total number of vehicles), and industrial structure (total shares of the industrial and service sectors)

- The frontier identifies the benchmark of best practice, which is the amount of energy a province would consume if it were operating efficiently, given its economic and structural characteristics

- As discussed by Zhang and Adom (2018), the selected variables help explain the scale of required energy inputs, and these variables indicate how much energy would be needed in the absence of inefficiency.

Determinant of inefficiencies

- Inefficiency term u_{it} captures the extent to which a province's energy consumption exceeds the minimum required level.
- Determinants include: foreign direct investment, human capital (proxied by average education), GDP per capita, urbanization, and policy implementation capacity (proxied by the size of green parkland), and energy price.

- Inclusion of energy price in the frontier function captures substitution or conservation effects in cost-minimizing behavior.
- Its inclusion in the inefficiency term reflects behavioral or institutional responses that lead to deviations from the efficient frontier.
- If price is highly regulated and uniform across provinces, its inclusion in the inefficiency may dominate the frontier effect.

Estimation

- To test the joint significance of the six variables in the inefficiency term, we estimated the model under the alternative hypothesis.
- The optimal order of the Bernstein polynomials was determined to be 4 based on BIC.

Table 1: Estimates of Bernstein polynomial coefficients and individual significance.

Variable	Bernstein coefficients					F stats	p-value
FDI	1.8505	0.9719	0.4609	2.5939	2.7350	2.2886	0.874
Human capital	1.4088	5.3167	0.2965	2.2574	1.5465	7.1013	0.030
GDP per capita	1.5600	6.6566	1.6660	1.6166	3.3575	2.3666	0.842
Urbanization	1.5530	1.3068	0.7733	4.0921	2.8213	2.9328	0.776
Policy	0.1252	0.8341	1.0341	2.2836	0.2883	2.3854	0.940
Energy price	0.4967	0.7936	2.1269	0.4859	2.0065	8.0805	0.018

Estimation and testing

- Based on the estimates of $\hat{\beta}$, the Bernstein coefficients, and the variance parameters $\hat{\sigma}_\eta^2$ and $\hat{\sigma}_v^2$, we computed the estimates of δ_i, for $i = 1, 2, \ldots, N$, using the approach outlined in Jondow et al. (1982) and Greene (2005).
- The unexplained inefficiencies $\hat{\delta}_i$ as a percentage of the overall inefficiencies $\hat{u}_{it}$ are around 0.1% to 0.5%.
- The joint test has a p-vale of 0.002, indicating that the 6 determinant variables are jointly significant.
- Testing results for marginal significance show that "human capital" and "energy price".

China's regulation of energy

- 2003–2010s: Coal and oil products moved toward market linkages; electricity tariffs remained administratively set but began separating wires (regulated whole sale) from energy (retail market).
- 2011–2016: Gas pricing pivoted to city-gate reform; oil pricing rules were tightened and made more responsive (with a 2016 floor).
- 2015 onward: Document #9 anchored a market-based wholesale power push; by 2021, coal-power prices were largely market-set with wider floating bands to pass through fuel costs–especially during the power crunch.
- 2021: full marketization of coal-fired on-grid tariffs. Abolished fixed benchmark pricing; all coal power traded via the market, with a $\pm 20\%$ floating band (no upper limit for energy-intensive users).

Renewable energy prices

- 2009–2018: Feed-in tariffs (FITs) for wind and solar set by NDRC, differentiated by resource zones, with subsidies funded via a renewable energy surcharge.
- From 2019: Shifted toward competitive auctions, lowering subsidy levels and integrating renewables into emerging power markets.

Policy implication

- Deregulation of energy price
- Promoting renewable energy

Conclusion

- Maximised Monte Carlo test for the significance of inefficiency determinants (joint and marginal tests)
- Inefficiencies are approximated by Bernstein polynomials
- Our test is implemented by carrying out the associated testing for point hull hypothesis among feasible regions

Future research

- Heterogeneous random effects or fixed effects among individuals
- Different Bernstein orders for different variables