ETF2700/ETF5970 Mathematics for Business

Lecture 6

Monash Business School, Monash University, Australia

Outline

Last week:

- Increasing/decreasing and convex/concave functions
- Single-variable optimization
- Linear and quadratic approximations
- Elasticity

This week:

- Functions of multiple variables
- Partial differentiation
- Slope of an iso curve

Relationship between variables

- x is the 'input' real-valued variable
- y is the 'output' real-valued variable

$$y = f(x), \quad x \in D$$

where

- D is a set of all possible inputs
- $f(x)$ is the 'output' real value assigned to each real-valued input $x \in D$

Derivative of a continuous function $f(x)$

The first principle says that

$$f'(x) = \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x)}{\Delta}$$

1) Rewrite $\frac{f(x+\Delta)-f(x)}{\Delta}$ without Δ in the denominator
2) Plug in $\Delta = 0$ to obtain the derivative

Second-order derivative: Derivative of a derivative

Example

- Linear: $f(x) = mx + c, f'(x) = m$
- Quadratic: $f(x) = ax^2 + bx + c, f'(x) = 2ax + b$
- Exponential: $f(x) = a^x, f'(x) = a^x \ln(a)$

The second-order derivative of $f(x)$:

$$f''(x) = \lim_{\Delta \to 0} \frac{f'(x + \Delta) - f'(x)}{\Delta}$$

$$f(x) \quad \xrightarrow{\text{derivative}} \quad f'(x) \quad \xrightarrow{\text{derivative}} \quad f''(x)$$

Example

- Linear: $f(x) = mx + c, f''(x) = 0$
- Quadratic: $f(x) = ax^2 + bx + c, f''(x) = 2a$
- Exponential: $f(x) = a^x, f''(x) = a^x (\ln(a))^2$

Function of two variables

- x is the first 'input' variable
- y is the second 'input' variable
- z is the 'output' variable

$$z = f(x, y), \quad (x, y) \in D$$

where

- D is a set of all possible input combinations
- $f(x, y)$ is the 'output' real value assigned to each real-valued input $(x, y) \in D$

Example: A market of two firms

Two firms in the market both produce cell phones.

- Competitor's price (in hundred dollars): $P_1 \in (0, \infty)$
- Your company's (in hundred dollars): $P_2 \in (5, 20)$
- Not identical: different designs, brands,...

Market demand function

- P_1: competitor's price is the first 'input' variable
- P_2: your company's price is the second 'input' variable
- Q_d: market demand is the 'output' variable

$$Q_d = f(P_1, P_2), \quad P_1 \in (5, 20) \text{ and } P_2 \in (0, \infty)$$

where we can specify

$$f(P_1, P_2) = 82 + P_1 - 3P_2$$

- Slope with respect to P_1: Fixing P_2, f is a linear function in P_1 with slope

$$m = \frac{f(P_1 + \Delta, P_2) - f(P_1, P_2)}{\Delta} = 1$$

- Slope with respect to P_2: Fixing P_1, f is a linear function in P_2 with slope

$$m = \frac{f(P_1, P_2 + \Delta) - f(P_1, P_2)}{\Delta} = -3$$

Slopes of $f(P_1, P_2)$? Another possible slope?

$$m = \frac{f(P_1 + a\Delta, P_2 + b\Delta) - f(P_1, P_2)}{\Delta} = a - 3b$$

for some real-valued a and b.

Partial derivative with respect to x

Consider the a function $f(x, y)$, for $(x, y) \in D$. Partial derivative of $f(x, y)$ with respect to (w.r.t.) x is

$$\frac{\partial f(x, y)}{\partial x} = \lim_{\Delta \to 0} \frac{f(x + \Delta, y) - f(x, y)}{\Delta}.$$

We may also write $f_x(x, y)$ or $f_1(x, y)$.

1) Fix y: Let $f(x, y) = g(x)$ which is a function of x only
2) Calculate $\frac{\partial f(x,y)}{\partial x} = g'(x)$.

Example

$$f(P_1, P_2) = 82 + P_1 - 3P_2$$

Determine the partial derivative of $f(P_1, P_2)$ w.r.t. P_1

1) Treat P_2 as a constant rather than a variable

$$f(P_1, P_2) = g(P_1) = mP_1 + c$$

with $m = 1$ and $c = 82 - 3P_2$

2) Derivative of $f(P_1, P_2)$ w.r.t. P_1 is the slope $g(P_1)$:

$$\frac{\partial f(P_1, P_2)}{\partial P_1} = g'(P_1) = m = 1$$

Partial derivative with respect to y

Consider the a function $f(x, y)$, for $(x, y) \in D$. Partial derivative of $f(x, y)$ w.r.t. y is

$$\frac{\partial f(x, y)}{\partial y} = \lim_{\Delta \to 0} \frac{f(x, y + \Delta) - f(x, y)}{\Delta}$$

Partial derivative with respect to y

We may also write $f_y(x, y)$ or $f_2(x, y)$.

1) Fix x: Let $f(x, y) = h(y)$ which is a function of y only
2) Calculate $\frac{\partial f(x,y)}{\partial y} = h'(y)$.

Example

$$f(P_1, P_2) = 82 + P_1 - 3P_2$$

Determine the partial derivative of $f(P_1, P_2)$ w.r.t. P_2.

1) Treat P_1 as a constant rather than a variable

$$f(P_1, P_2) = h(P_2) = mP_2 + c$$

with $m = -3$ and $c = 82 + P_1$.

2) Derivative of $f(P_1, P_2)$ w.r.t. P_2 is the slope $h(P_2)$:

$$\frac{\partial f(P_1, P_2)}{\partial P_2} = h'(P_2) = m = -3$$

Other possible slopes?

We can show that

$$\lim_{\Delta \to 0} \frac{f(x + a\Delta, y + b\Delta) - f(x, y)}{\Delta} = \frac{\partial f(x, y)}{\partial x} \cdot a + \frac{\partial f(x, y)}{\partial y} \cdot b$$

Partial derivatives are enough to describe the class of slopes

Approximation with partial derivative

$$f_x(x, y) = \lim_{\Delta \to 0} \frac{f(x + \Delta, y) - f(x, y)}{\Delta}$$

For $\Delta \approx 0$

$$f_x(x, y) \approx \frac{f(x + \Delta, y) - f(x, y)}{\Delta}$$

Substitute $\Delta = 1$ (assuming Δ is approximately 0) and obtain

$$f_x(x, y) \approx f(x + 1, y) - f(x, y)$$

Interpret partial derivatives ($\Delta = 1$)

$$f_x(x, y) \approx f(x+1, y) - f(x, y)$$

- Holding y constant, one unit increase in x from the point (x, y) implies approximately a change in f by $f_x(x, y)$ unit.
- A similar argument applies to $f_y(x, y)$.

Example

Interpret the partial derivative of

$$g(x, y) = x^2 + y$$

w.r.t. x at $(3, 5)$ from the view of approximation.

1) Treat y as a constant, so

$$g_x(x, y) = 2x + 0 = 2x \;\Rightarrow\; g_x(3, 5) = 6$$

2) Holding y constant, one unit increase in x from the point $(3, 5)$ implies approximately 6 unit changes in $g(x, y)$.

True increment: $g(4, 5) - g(3, 5) = 7$

Differential: Simultaneous changes

Let x and y be changed by $dx \approx 0$ and $dy \approx 0$.

$$f(x + dx, y + dy) - f(x, y) \approx f'_x(x, y) \cdot dx + f'_y(x, y) \cdot dy$$

If we use $df(x, y)$ to denote the change in f:

$$df(x, y) = f(x + dx, y + dy) - f(x, y)$$

we have

$$df(x, y) = f'_x(x, y) \cdot dx + f'_y(x, y) \cdot dy.$$

It is called the differential of f in (x, y) for the change (dx, dy), or sometimes just the differential of $f(x, y)$.

Example

$$g(x, y) = x + y^2, \quad x, y \in (-\infty, \infty)$$

1. Treat y as a constant rather than a variable:

$$g'_x(x, y) = 1 + 0 = 1$$

2. Treat x as a constant rather than a variable:

$$g'_y(x, y) = 0 + 2y = 2y$$

The differential of $g(x, y)$ is

$$dg(x, y) = 1 \cdot dx + 2y \cdot dy$$

Partial Elasticity

For a single variable function $f(x)$, its elasticity at point x is

$$\mathrm{El}_x f(x) = \frac{f'(x)x}{f(x)}$$

Interpret elasticity

Increasing x by 1% from point x implies approximately $\mathrm{El}_x f(x)$% changes in $f(x)$.

Partial Elasticity

Partial elasticity of $f(x, y)$ w.r.t. x and y at point (x, y) are, respectively,

$$\mathrm{El}_x f(x, y) = \frac{f_x(x, y)x}{f(x, y)}, \quad \mathrm{El}_y f(x, y) = \frac{f_y(x, y)y}{f(x, y)}$$

Interpret partial elasticity with respect to x

Holding y constant, increasing x by 1% from point (x, y) implies approximately $\mathrm{El}_x f(x, y)$% changes in $f(x, y)$.

Partial price elasticity of demand

Determine the partial elasticity of

$$f(P_1, P_2) = 82 + P_1 - 3P_2$$

with respect to P_1 at point $(3, 5)$

- We know the partial derivative is $f_1(P_1, P_2) = 1$, so $f_1(3, 5) = 1$
- By definition

$$\text{El}_{P_1} f(3, 5) = \frac{f_x(3, 5) \cdot 3}{f(3, 5)} = \frac{1 \cdot 3}{70} = \frac{3}{70}$$

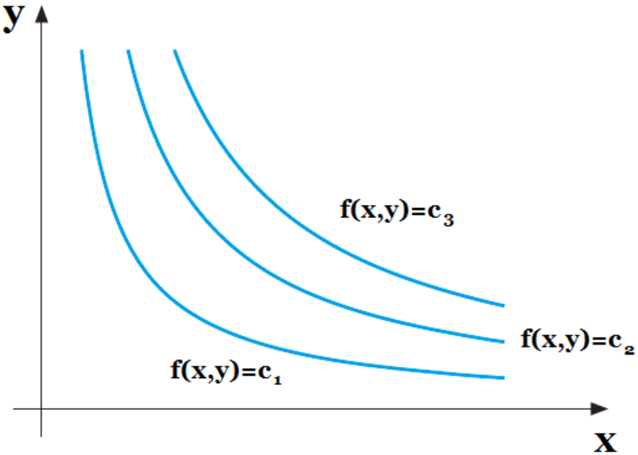
Iso curve (level curve)

Consider a function $f(x, y)$, for $(x, y) \in D$. An iso curve, or level curve, of $f(x, y)$ consists of the points satisfying the equation

$$f(x, y) = c \quad \text{or} \quad f(x, y) - c = 0$$

for some known value of c, such that $(x, y) \in D$.

Iso curves: Illustration



Implicit Function

- Consider a particular iso curve $f(x, y) = c$.
  1) x is an 'input' variable (in the domain)
  2) y is the only y in the domain that satisfies $f(x, y) = c$
- We may then define a function g such that $y = g(x)$.
- For a fixed value of x, we can find a value of y, such that $f(x, y) = c$. In this sense, y is a function of x.
- The function g is called the implicit function for y defined by the iso curve $f(x, y) = c$.

Example: Market demand function

$$f(P_1, P_2) = 82 + P_1 - 3P_2, \quad P_1 \in (0, \infty), P_2 \in (5, 20)$$

Consider the point $(P_1, P_2) = (3, 5)$, where $f(3, 5) = 70$.
The iso curve passing through $(3, 5)$ is $f(P_1, P_2) = 70$
$\Rightarrow 82 + P_1 - 3P_2 = 70 \quad \Rightarrow P_2 = \frac{1}{3}P_1 + 4$
The corresponding implicit function $g(P_1) = \frac{1}{3}P_1 + 4$.

Slope of an Iso curve

Consider an iso curve with implicit function $g(x)$ such that

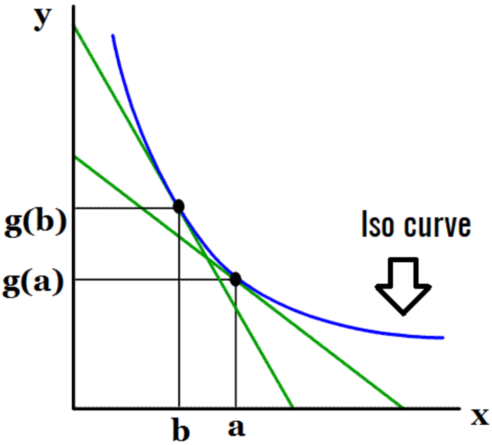
$$f(x, g(x)) = c$$

The slope of the above iso curve at some point $(a, g(a)) \in D$ (on this iso curve) is $g'(a)$.

Example of demand:

- The slope of the iso curve $f(P_1, g(P_1)) = 70$ is $g'(P_1) = 1/3$, which is the slope of the implicit function.
- Negative slope, which is $-g'(a)$, is called the marginal rate of substitution of y for x at the point $(a, g(a))$.
- The variable y needs to be changed from $g(a)$ by approximately $g'(a)$ units for each unit increase in x from a to maintain the same $f(x, y) = f(a, g(a))$.

Slope of an Iso curve: Illustration



Market demand: $f(P_1, P_2) = 82 + P_1 - 3P_2$

- The implicit function for the iso curve passing through the point $(3, 5)$ is

$$P_2 = g(P_1) = \frac{1}{3}P_1 + 4 \quad \text{with} \quad g'(P_1) = \frac{1}{3}$$

  which gives $g'(3) = \frac{1}{3}$.
- Our price (P_2) need to increase from 5 hundred dollars by approximately $1/3$ hundred dollars for each hundred dollars increase in the competitor's price from 3 hundred dollars, in order to maintain the same market demand $f(3, 5) = 70$ millions.

Market Equilibrium

Suppose your company's supply of the cell phone is given by

$$Q_s = -10 + P_2^2$$

and the current prices are $P_1 = 16$ and $P_2 = 9$.

- Show that $(P_1, P_2) = (16, 9)$ is a (possible) market equilibrium,that is, $Q_d = Q_s$:
  $Q_d = 82 + P_1 - 3P_2 = 82 + 16 - 3 \cdot 9 = 71.$
  $Q_s = -10 + 9^2 = -10 + 81 = 71.$
- The current market is in equilibrium, i.e. $Q_d = Q_s$, and the current prices are $(P_1, P_2) = (16, 9)$.

Question: If the competitor's price P_1 increases by 1 hundred dollars, by how much should our price P_2 increase to maintain market equilibrium?

Slope in market equilibrium

In market equilibrium we have $Q_d = Q_s$, that is

$$82 + P_1 - 3P_2 = -10 + P_2^2$$

In other words, (P_1, P_2) is on the iso curve of the function

$$f(P_1, P_2) = -P_2^2 - 3P_2 + P_1 + 92$$

associated with the equation $f(P_1, P_2) = 0$.

Find out the implicit function?

For each given P_1, we need to solve the equation

$$-P_2^2 - 3P_2 + P_1 + 92 = 0$$

Solve by the abc method

$$\Delta = (-3)^2 - 4 \cdot (-1) \cdot (P_1 + 92) = 377 + 4P_1 > 0$$

$$x_1 = \frac{3 - \sqrt{377 + 4P_1}}{-2}, \quad x_2 = \frac{3 + \sqrt{377 + 4P_1}}{-2} < 0.$$

Therefore, $P_2 = g(P_1) = (\sqrt{377 + 4P_1} - 3)/2$.

In equilibrium, we have

$$P_2 = g(P_1) = \frac{\sqrt{377 + 4P_1} - 3}{2}$$

which implies taht

$$g'(P_1) = \frac{1}{\sqrt{377 + 4P_1}}$$

The derivative of the iso curve at $(P_1, P_2) = (16, 9)$ is

$$g'(16) = \frac{1}{\sqrt{377 + 4 \cdot 16}} = \frac{1}{21}$$

Our price P_2 need to increase approximately $\frac{1}{21} \times 100 \approx \$4.76$ to maintain market equilibrium.
True increment in our price is: $(g(17) - g(16)) \times 100 \approx \$4.75$

Approximation with slopes

Is there an easier way to determine $g'(16)$?

Recall that

$$\lim_{\Delta \to 0} \frac{f(x + a\Delta, y + b\Delta) - f(x, y)}{\Delta}$$
$$= f_x(x, y) \cdot a + f_y(x, y) \cdot b$$

If let $\Delta = 1$ (assuming Δ is approx 0), then

$$f(x + a, y + b) - f(x, y) \approx f_x(x, y) \cdot a + f_y(x, y) \cdot b$$

An easier way to determine the slope

- Suppose $(P_1, P_2) = (16, 9)$ is changed to another equilibrium point $(16 + 1, 9 + \Delta)$. Then

$$\begin{aligned} & f(16+1, 9+\Delta) - f(16, 9) \\ \approx & f_1(16, 9) \cdot 1 + f_2(16, 9) \cdot \Delta \end{aligned}$$

- The terms on left-hand-side are ended with 0 due to equilibrium:

$$0 \approx f_1(16, 9) \cdot 1 + f_2(16, 9) \cdot \Delta$$

- Rearrange to obtain $\Delta \approx -\frac{f_1(16,9)}{f_2(16,9)}$. We also know $\Delta \approx g'(16)$.
- Could $g'(16) = -\frac{f_1(16,9)}{f_2(16,9)}$? Yes
- On the iso curve $f(x, y) = c$, where the implicit function is $y = g(x)$, the derivative of $g(\cdot)$ can be derived by the negative ratio of partial derivatives of $f(x, y)$.

Implicit differentiation

Note that $f(P_1, P_2) = -P_2^2 - 3P_2 + P_1 + 92$ gives

$$f_1(P_1, P_2) = 1 \text{ and } f_2(P_1, P_2) = -2P_2 - 3$$

We have

$$g'(16) = -\frac{f_1(16,9)}{f_2(16,9)} = -\frac{1}{-2 \cdot 9 - 3} = \frac{1}{21}.$$

This approach is called implicit differentiation

Implicit differentiation: General

Consider a particular iso curve of function $f(x, y)$ with implicit function $g(x)$ such that

$$f(x, g(x)) = c$$

The slope of the above iso curve at point (x, y) is

$$g'(x) = -\frac{f_x(x,y)}{f_y(x,y)}$$

Summary

Function of two variables:

- Partial derivative, and partial elasticity
- Approximations
- Iso curve and its slopes
- Implicit Differentiation

Function of three variables

- x is the first 'input' variable
- y is the second 'input' variable
- λ is the third 'input' variable
- z is the 'output' variable

$$z = f(x, y, \lambda), \quad (x, y, \lambda) \in D$$

where

- D is a set of all possible input combinations
- $f(x, y, \lambda)$ is the output value assigned to each input vector

Partial derivative

The partial derivative of $f(x, y, \lambda)$ with respect to x is

$$\frac{\partial f(x, y, \lambda)}{\partial x} = \lim_{\Delta \to 0} \frac{f(x + \Delta, y, \lambda) - f(x, y, \lambda)}{\Delta}.$$

We may also write $f_x(x, y, \lambda)$ or $f_1(x, y, \lambda)$.

1) Fix y and λ: $f(x, y, \lambda) = g(x)$ only a function of x
2) Calculate $\frac{\partial f(x,y,\lambda)}{\partial x} = g'(x)$.

Similarly we can define $f_y(x, y, \lambda)$ and $f_\lambda(x, y, \lambda)$

Example

Determine the partial derivative

$$f(x, y, \lambda) = 2x + y + \lambda(x^2 + y^2 - 1)$$

with respect to x.

1) Treat y and λ as constants, then

$$f(x, y, \lambda) = g(x) = ax^2 + bx + c$$

   with $a = \lambda$, $b = 2$ and $c = y + \lambda(y^2 - 1)$

2) $f_x(x, y, \lambda) = g'(x) = 2ax + b = 2\lambda x + 2$

Example

Determine the partial derivative

$$f(x, y, \lambda) = 2x + y + \lambda(x^2 + y^2 - 1)$$

with respect to λ.

1) Treat x and y as constants, then

$$f(x, y, \lambda) = g(\lambda) = m\lambda + c$$

with $m = x^2 + y^2 - 1$ and $c = 2x + y$

2) $f_\lambda(x, y, \lambda) = g'(\lambda) = m = x^2 + y^2 - 1$